

Local algebra and string theory.

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Abstract

The $\beta\gamma$ system on the cone of pure spinors is an integral part of the string theory developed by N. Berkovits.

This $\beta\gamma$ system offer a number of questions for pure mathematicians: what is a precise definition of the space of states of the theory? Is there a mathematical explanations for various dualities (or pairings) predicted by physicists? Can a formula for partition function be written? To help me to answer these questions I use local algebra in an essential way.

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1 Introduction

The notion of a $\beta\gamma$ system is not very familiar to mathematicians. Let us go over some of its basic aspect before we turn to detailed mathematical analysis of $\beta\gamma$ system, whose target is the cone of pure spinors. The interested reader can consult [38] for a more comprehensive account.

Let $\Lambda = \bigoplus_k \Lambda^k$ stands for the exterior algebra on some vector space or a bundle. I fix a smooth complex manifold \mathcal{X} of real dimension $2n$, which will be the target of the $\beta\gamma$ system.

Let $T_{\mathbb{R}}^*$ be the real cotangent bundle of \mathcal{X} . Complexification of $T_{\mathbb{R}}^*$ splits into the direct sum $T_{\mathbb{C}}^* = T^* + \overline{T}^*$, which leads to the (p, q) -decomposition $\Lambda^k T_{\mathbb{C}}^* = \bigoplus_{p+q=k} \Lambda^p T^* \otimes \Lambda^q \overline{T}^*, k = 0, \dots, 2n$. The de Rham differential splits accordingly: $d = \partial + \bar{\partial}$.

Let Σ be a complex one-dimensional manifold, considered as a C^∞ -surface. The fields of the $\beta\gamma$ system are:

1. a smooth map $\beta : \Sigma \rightarrow \mathcal{X}$,
2. a smooth section γ of the pullback $\beta^* T^* \otimes T_{\Sigma}^*$.

The anti holomorphic part $\bar{\partial}\beta$ of the differential $D\beta$ is a section of $\beta^*T^* \otimes \bar{T}_\Sigma^*$. The pairing $\langle \bar{\partial}\beta, \gamma \rangle$ is a $(1,1)$ -form on Σ . The action of the theory

$$S(\beta, \gamma) = \int_\Sigma \langle \bar{\partial}\beta, \gamma \rangle$$

makes sense when, for example, γ has a compact support.

Let G be a group of holomorphic symmetries of \mathcal{X} . Elements of the space $\mathcal{M}aps_{hol}(\Sigma, G)$ of holomorphic maps are symmetries of S . The functional S is also invariant with respect to (infinitesimal) holomorphic reparametrizations of Σ .

Points of the space of solutions \mathcal{M}_Σ of the Euler-Lagrange equations

$$\bar{\partial}\beta = 0$$

$$\bar{\partial}\gamma = 0$$

are a holomorphic map $\beta : \Sigma \rightarrow \mathcal{X}$ and a holomorphic section γ of $\beta^*T^* \otimes T_\Sigma^*$.

The space $\mathcal{M}_{\mathbb{C}^\times}$, or the phase space, is equipped with a "pdq"-type form α . Its value on $\delta\beta$ at $(\beta, \gamma) \in \mathcal{M}_{\mathbb{C}^\times}$ is equal to

$$\alpha(\delta\beta) = \oint \langle \gamma, \delta\beta \rangle.$$

Exterior variational derivative of α is a closed two-form $(\delta\beta, \delta\gamma) = \oint \langle \delta\beta, \delta\gamma \rangle$.

By the Noether theorem, the semidirect product

$$Vect_{poly}(\mathbb{C}^\times) \ltimes \mathcal{M}aps_{poly}(\mathbb{C}^\times, \mathfrak{g}), \quad \mathfrak{g} = \text{Lie}(G) \quad (1.1)$$

of the Lie algebras is acting on the phase space $\mathcal{M}_{\mathbb{C}^\times}$. It is also should be acting projectively on the space of quantum states.

In the field theory, the space $\mathcal{M}_{\mathbb{C}^\times}$ plays the role of the cotangent bundle to

$$\mathcal{Z}_{\mathbb{C}^\times} = \{\beta : \mathbb{C}^\times \rightarrow \mathcal{X} | \bar{\partial}\beta = 0\} \subset \mathcal{M}_{\mathbb{C}^\times}.$$

1.1 Quantization

Before I discuss quantization of the $\beta\gamma$ -system, I need to remind the reader some of its general aspects. Let (Y, g_{ij}) be a finite-dimensional Riemannian manifold, thought as a configuration space of a quantum mechanical system with constraints. The Hilbert space obtained by quantizing the phase space $T_{\mathbb{R}, Y}^*$ is the L^2 space. It is the closure of the space of the complex compactly supported smooth functions $C_{comp}^\infty(Y)$ with respect to the inner product

$$(a, b) = \int_Y a \bar{b} dvol_g, \quad a, b \in C^\infty(Y). \quad (1.2)$$

The space $C_{comp}^\infty(Y)$ is a module over the algebra of differential operators on Y (a D-module for short). There are other D-modules which could be candidates in quantum Hilbert spaces. An important example of a D-module is the space of distributions D_U with the support on a submanifold $U \subset Y$. It is a common believe however, that distributions never form a space with an inner product. The reason - the integrand in (1.2) for distributions a, b is ill-defined. Still, there is an alternative way to define the pairing. Let U_+, U_- be two closed submanifolds of complementary dimension that intersect transversally at points x_1, \dots, x_k . $I(Y, U_\pm)$ is the space of conormal distributions in the sense of Hörmander [31]. There is a natural map $I(Y, U_+) \otimes I(Y, U_-) \xrightarrow{m} I(Y, \{x_1, \dots, x_n\})$ - the exterior product of distributions ([46], Corollary 3, p. 113). I can use this map to define a pairing between spaces $I(Y, U_+) \otimes I(Y, U_-)$ by the formula

$$\langle a, b \rangle = \int_Y m(a, b) dx.$$

In the presence of a symmetry

$$\sigma : Y \rightarrow Y \text{ such that } \sigma(U_\pm) = U_\mp,$$

the pairing can be turned into an inner product

$$(a, b) = \langle a, \sigma^* \bar{b} \rangle.$$

The theory of distributions of S. Sobolev and L. Schwartz has cohomological generalization in M.Sato's works on hyperfunctions (see e.g. [42] for the account). Hyperfunctions are defined on a real analytic space Y embedded into its Grauert tube $Y^\mathbb{C}$. The elements of local cohomology $H_Y^{\dim Y}(Y^\mathbb{C}, \mathcal{O})$ of the sheaf of analytic functions \mathcal{O} on $Y^\mathbb{C}$ with support on Y are the Sato's hyperfunctions. A typical example of a hyperfunction is Cauchy principal value $f(x) \rightarrow \text{p.v.} \int \frac{f(x)}{x} dx$. In the same vein, one can define complex-analytic hyperfunctions on a complex manifold Z with support on a complex submanifold

$$Z' \subset Z. \tag{1.3}$$

The basic example of Z' is a divisor of a rational function $g : \mathbb{C} \rightarrow \mathbb{C}$. It define a functional on the space of analytic functions by the formula

$$f \rightarrow \oint f(z)g(z)dz$$

The integration is taken over a circle of sufficiently large radius. The cohomological nature of hyperfunctions is already seen in this simple example. The integral is invariant under substitution $g(z) \rightarrow g(z) + h(z)$ where $h(z)$ is a polynomial. Thus the space of hyperfunctions is in fact the first cohomology of the Cousin complex $\mathcal{O}[\mathbb{C}] \hookrightarrow \mathcal{O}[\mathbb{C} \setminus Z']$. $\mathcal{O}[\mathbb{C} \setminus Z']$ is the space of rational functions with poles at Z' .

The space of hyperfunctions is a D-module. If σ is a holomorphic involution such that $\sigma(Z') \cap Z'$ is discrete and transversal, then the pairing on local cohomology is the composition

$$H_{Z'}^i(Z, \mathcal{O}) \otimes H_{Z'}^{\dim Z - i}(Z, \mathcal{O}) \xrightarrow{\mathfrak{m}} H_{\sigma(Z') \cap Z'}^{\dim Z}(Z, \mathcal{O}) \xrightarrow{\text{res}} \mathbb{C}, \quad i = \dim Z/2.$$

The map \mathfrak{m} is a cup product (see e.g. [28] Section II.10), res is the Poincaré residue. This construction will be implemented in this paper.

1.2 Regularization

Let us return to $\beta\gamma$ systems. I would like to specialize the target to a conical set. To be more precise, let $x^i, i = 1, \dots, n$ be the coordinates of a vector e in the standard basis for \mathbb{C}^n . Homogeneous algebraic equations

$$r^p(x^1, \dots, x^n), p = 1, \dots, k \quad (1.4)$$

define a possibly nonsmooth algebraic cone \mathcal{X} in \mathbb{C}^n . Though $\bar{\partial}$ is undefined on the whole \mathcal{X} , the spaces Z_Σ do make sense as

$$Z_\Sigma = \{\beta : \Sigma \rightarrow \mathbb{C}^n | \bar{\partial}\beta = 0, \beta(z) \in \mathcal{X} \forall z \in \Sigma\}. \quad (1.5)$$

It is a common practice in quantum field theory to approximate an infinite-dimensional Z_Σ by more manageable finite-dimensional spaces. This procedure is called regularization.

The finite-dimensional approximations $Z_{\mathbb{C}^\times}(N, N') \subset Z_{\mathbb{C}^\times}, N \leq N'$ consists of the maps

$$e(z) = \sum_{s=1}^n x^s(z) e_s = \sum_{N \leq k \leq N'} \sum_{i=1}^n x^{s^k} z^k e_s \quad (1.6)$$

from \mathbb{C}^\times to \mathcal{X} (s^k is a multi-index) which are Laurent polynomials in z . The algebra of polynomial functions on $Z(N, N')$ is the quotient of $\mathbb{C}[x^{s^k}], s = 1, \dots, n, k = N, \dots, N'$. The ideal is generated by relations $R^{p^k}(x)$ whose generating function satisfies

$$\sum_{k=\deg(r^p)N}^{\deg(r^p)N'} R^{p^k}(x) z^k = r^p(x^1(z), \dots, x^n(z)).$$

In our application, the regularized pair (1.3) is

$$Z(0, N') \subset Z(N, N').$$

The cone \mathcal{X} has a singularity at the origin. This singularity propagates in $Z(N, N')$ and makes the latter highly singular. This makes the definition of conormal distributions $I(Z, Z')$ problematic. At this

point the algebraic version of hyperfunctions $H_{Z'}^{\text{codim} Z'}(Z, \mathcal{O})$ comes to rescue because this sheaf-theoretic definition is less sensitive to singularities of the space.

Technically it is more convenient to replace the space of analytic maps (1.5) by the space of polynomial maps

$$Z_{\mathbb{C}^\times}^{\text{poly}} := \bigcup_{N, N'} Z_{\mathbb{C}^\times}(N, N'),$$

$$Z_{\mathbb{C}}^{\text{poly}} := \bigcup_{N'} Z_{\mathbb{C}}(0, N').$$

The symmetry group $Z_{\mathbb{C}^\times}^{\text{poly}}$ always contains the product $\mathbb{C}^\times \times \mathbb{C}^\times$. The first factor corresponds to dilations of the cone $Z_{\mathbb{C}^\times}^{\text{poly}}$. The second factor, which will be denoted by \mathbf{T} , corresponds to the loop rotation. Let $w = (a, u)$ be a $\mathbb{C}^\times \times \mathbf{T}$ weight. In the following V^w stands for weight w subspace in representations V of $\mathbb{C}^\times \times \mathbf{T}$.

Conjecture 1 *Let us assume that the algebra of global algebraic functions $\mathbb{C}[\mathcal{X}]$ is Gorenstein (2.36). Let \mathbf{P} stands for projectivization of the cone. Suppose that $X = \mathbf{P}(\mathcal{X})$ is smooth Fano such that $\text{ind} X^1$ is large (at least $\text{ind} X > 1$). Then*

1. $\mathbb{C}[Z(N, N')]$ is Gorenstein for any $N < N'$.
2. One can define a limit of weight spaces

$$H_{Z_{\mathbb{C}}^{\text{poly}}}^{i+\frac{\infty}{2}}(Z_{\mathbb{C}^\times}^{\text{poly}}, \mathcal{O})^w := \varinjlim_N \varprojlim_{N'} H_{Z(0, N')}^{i+\text{codim} Z(0, N')}(Z(N, N'), \mathcal{O})^w, \quad (1.7)$$

when $N \rightarrow -\infty, N' \rightarrow \infty$. $H_{Z_{\mathbb{C}}^{\text{poly}}}^{i+\frac{\infty}{2}}(Z_{\mathbb{C}^\times}^{\text{poly}}, \mathcal{O}) := \bigoplus_w H_{Z_{\mathbb{C}}^{\text{poly}}}^{i+\frac{\infty}{2}}(Z_{\mathbb{C}^\times}^{\text{poly}}, \mathcal{O})^w$

3. There is a nondegenerate pairing

$$H_{Z_{\mathbb{C}}^{\text{poly}}}^{i+\frac{\infty}{2}}(Z_{\mathbb{C}^\times}^{\text{poly}}, \mathcal{O}) \otimes H_{Z_{\mathbb{C}}^{\text{poly}}}^{k-i+\frac{\infty}{2}}(Z_{\mathbb{C}^\times}^{\text{poly}}, \mathcal{O}) \rightarrow \mathbb{C} \quad (1.8)$$

for some k depending on \mathcal{X} . In all known examples $k = \text{coind}(\mathbf{P}(\mathcal{X})) + 1$.

4. The groups $H_{Z_{\mathbb{C}}^{\text{poly}}}^{i+\frac{\infty}{2}}(Z_{\mathbb{C}^\times}^{\text{poly}}, \mathcal{O})$ are representations with weights bounded from below of a central extension of (1.1). The pairing (1.8) is compatible with the symmetries.
5. I conjecture that $\dim H_{Z(0, N')}^{i+\text{codim} Z(0, N')}(Z(N, N'), \mathcal{O})^w \leq C_w$. The constant C_w doesn't depend on N and N' .

¹Index $\text{ind} X$ of a projective Fano manifold X is a maximal integer such that $\mathcal{K} = \mathcal{L}^{\otimes \text{ind} X}$ for the canonical \mathcal{K} and some line bundle \mathcal{L} . $\text{coind} X = \dim X - \text{ind} X$.

There is a conjecture closely related to item (5).

Conjecture 2 *I continue using assumptions of Conjecture 1. Fix a weight $w = (a, u)$. I conjecture that*

$$\dim_{\mathbb{C}} \text{Ext}_{\mathcal{O}[\mathcal{Z}(N, N')]}^l(\mathbb{C}, \mathbb{C})^w \leq C_{w, l}$$

where $C_{w, l}$ doesn't depend on N, N' .

Remark 1.9 *It is worthwhile to indicate where the maps $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O})$ in (1.7) can come from. There is a pairing between cohomology $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O})$ and $H_{\mathcal{Z}(N, -1)}^i(\mathcal{Z}(N, N'), \mathcal{O})$ when $\mathcal{Z}(N, N')$ is Gorenstein. The maps $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O}) \rightarrow H_{\mathcal{Z}(0, N'-1)}^i(\mathcal{Z}(N, N'-1), \mathcal{O})$ that form the inductive part of the limit, are the restriction map. The "wrong way" maps are $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O}) \rightarrow H_{\mathcal{Z}(0, N')}^{i+\text{codim}}(\mathcal{Z}(N+1, N'), \mathcal{O})$. They form the direct part of the limit, are the adjoint to the restriction maps $H_{\mathcal{Z}(N+1, -1)}^i(\mathcal{Z}(N+1, N'), \mathcal{O}) \rightarrow H_{\mathcal{Z}(N, -1)}^i(\mathcal{Z}(N, N'), \mathcal{O})$. In my treatment I will use a different but equivalent construction of these maps.*

Local cohomology groups (1.7) can be computed by taking sub-complex of sections $\Gamma_{\mathcal{Z}_{\mathbb{C}}(N')}(I^{\bullet})$ with the support on $\mathcal{Z}_{\mathbb{C}}(N')$ in an injective resolution I^{\bullet} of the sheaf \mathcal{O} . The total complex of the Koszul bicomplex

$$\mathcal{T}_I^k(N, N') := \bigoplus_{i-j=k} B_j(\Gamma_{\mathcal{Z}_{\mathbb{C}}(N')}(I^i), \{x^{s^k}\})$$

conjecturally has its own pairing in cohomology $BV^i(N, N') := H^i(\mathcal{T}_I(N, N'))$.

Conjecture 3 *Let n be as in (1.4).*

1. *One can define a limit of linear spaces $BV^{i+\frac{\infty}{2}} = \bigoplus_w BV^{i+\frac{\infty}{2}, w}$*

$$BV^{i+\frac{\infty}{2}, w} := \lim_{\substack{\rightarrow \\ N}} \lim_{\substack{\leftarrow \\ N'}} BV^{i+\text{codim}\mathcal{Z}(0, N') + Nn}(N, N')^w.$$

2. *There is a nondegenerate pairing*

$$BV^{i+\frac{\infty}{2}} \otimes BV^{k-i+\frac{\infty}{2}} \rightarrow \mathbb{C} \tag{1.10}$$

with k the same as in Conjecture 1.

In the following sections, I will prove Conjecture 1 (with omission of the Lie algebra action) and Conjecture 3 for the cone over the isotropic Grassmannian, which I will define presently.

1.3 The cone of pure spinors \mathcal{C}

In this section, I will briefly remind the reader the definition of the space of pure spinors. Let \mathring{V} be a ten-dimensional complex vector space, equipped with a nondegenerate complex-linear inner product (\cdot, \cdot) . The central object of study of this paper is an affine cone \mathcal{C} over $\text{OGr}^+(5, 10)$ - a connected component of the Grassmannian of five-dimensional isotropic subspaces. I refer the reader to [12],[15] and [17] for details on isotropic grassmannians.

The cone of pure spinors

$$\mathcal{C} \subset \mathring{S}_+ \quad (1.11)$$

over $\text{OGr}^+(5, 10)$ is an algebraic variety over \mathbb{C} in 16-dimensional linear space \mathring{S}_+ with coordinates

$$\begin{aligned} \lambda, p_i, w_{ij}, \\ 1 \leq i, j \leq 5, w_{ij} = -w_{ji}. \end{aligned} \quad (1.12)$$

\mathcal{C} is defined by equations

$$\begin{aligned} \lambda p_i - \text{Pf}_i(w) &= 0 \quad i = 1, \dots, 5, \\ pw &= 0. \end{aligned} \quad (1.13)$$

$\text{Pf}_i(w), 1 \leq i \leq 5$ are the principal Pfaffians of w .

Definition 1.14 \mathring{A} is the graded algebra generated by (1.12) that are subject to relations (1.13)

1.4 Formulation of the results

In this section I collected main theorems which are proved in this paper. In the end of the section the reader will find a brief synopsis of the remaining parts.

Denote

$$\begin{aligned} \mathcal{Z}_{\mathbb{C}^\times, \mathcal{C}}^{\text{poly}} \text{ by } \mathcal{Z}_{\mathbb{C}^\times}, \quad \mathcal{Z}_{\mathbb{C}, \mathcal{C}}^{\text{poly}} \text{ by } \mathcal{Z}_{\mathbb{C}}, \\ \mathcal{Z}_{\mathbb{C}^\times, \mathcal{C}}^{\text{poly}}(N, N') \text{ by } \mathcal{Z}(N, N'). \end{aligned} \quad (1.15)$$

Theorem 1.16 *Conjecture 1 (with the omission of the affine Lie algebra action) hold true for $\mathcal{X} = \mathcal{C}$.*

Proof. Item 1 follows from Proposition 2.35). Item 2 follows from Definition 6.26. Item 3 follows from Proposition 6.33. Item 4 is item 1 from Corollary 6.27, . Item 5 is item 2 from Corollary 6.27 . The constant k is equal to 3. All the structures that appear are compatible with the $\text{Spin}(10)$ -action. ■

Theorem 1.17 *Conjecture 2 hold true for $\mathcal{X} = \mathcal{C}$.*

Proof. Follows from Proposition 5.16. ■

Theorem 1.18 *Items 1,2 of Conjecture 3 hold true for $Z_{\mathbb{C}^\times}(N, N')$. The linear spaces are equipped with $\text{Spin}(10)$ -action.*

Proof. See (7.38), 7.41 in Section 7.4. ■

One of the observation of this paper is that there is a nontrivial Thom class $\text{th} \in \text{Ext}_P^8(A_N^{N'}, A_{N-1}^{N'})$ for Gorenstein algebras $A_N^{N'} = \mathcal{O}[Z(N, N')]$. P is the graded polynomial algebra on the same set of generators as $A_N^{N'}$. Product with th defines the maps

$$\text{th} : H_{Z(0, N')}^i(Z(N, N'), \mathcal{O}) \rightarrow H_{Z(0, N')}^{i+8}(Z(N+1, N'), \mathcal{O}). \quad (1.19)$$

Together with the pullbacks

$$H_{Z(0, N')}^i(Z(N, N'), \mathcal{O}) \rightarrow H_{Z(0, N'-1)}^i(Z(N, N'-1), \mathcal{O})$$

these maps turn $\{H_{Z(0, N')}^i(Z(N, N'), \mathcal{O})\}$ into a bidirect system (1.7).

Virtual characters $\mathbb{C}^\times \times \mathbf{T} \times \text{Spin}(10)$ is a subgroup in the groups of symmetries of $H^{i+\frac{\infty}{2}}$. Let $\tilde{\mathbf{T}}^5$ be the maximal torus of $\text{Spin}(10)$. Denote

$$\text{Aut} := \mathbb{C}^\times \times \mathbf{T} \times \tilde{\mathbf{T}}^5. \quad (1.20)$$

$\tilde{\mathbf{T}}^5$ is the subgroup of diagonal matrices $\tilde{\mathbf{T}}^5 \subset \widetilde{\text{GL}}(5) \subset \text{Spin}(10)$ (" \sim " stands for two-sheeted cover).

The virtual character function $Z(t, q) := Z(t, q, 1)$

$$Z(t, q, z) := \sum_{i=0}^3 (-1)^i \chi_{H^{i+\frac{\infty}{2}}}(t, q, z), (t, q, z) \in \text{Aut},$$

is understood as an element in $\mathbb{Z}((t))[[q]] \cap \mathbb{Q}(t)[[q]]$. It satisfies a pair of equations

$$Z(t, q) = -t^{-8} Z(1/t, q), \text{ field-antifield symmetry }, \quad (1.21)$$

$$Z(t, q) = -q^2 t^{-4} Z(q/t, q) \text{ *-conjugation symmetry.} \quad (1.22)$$

Equations were written for the first time in [2] (equations 4.40 and 4.41). I have an independent verifications (1.22) in Corollaries 6.34 and 7.43. Equation (4.120 4.121) contain a formula for a function $Z_{\mathbf{a}N}^{N'}$ whose limit $N \rightarrow -\infty, N' \rightarrow \infty$ is Z . First few terms of q -expansion are given in (4.122).

The space of states The space of states, as I define it in 6.17, might look rather mysterious. I give an elementary description of $H^{3+\frac{\infty}{2}}$ in Proposition 6.41.

The structure of the groups $H^{i+\frac{\infty}{2}}, i = 1, 2$ is obscure. Experimentations with *Macaulay2* show that $H_{z(0, N')}^{i+\text{codim} z(0, N')}(\mathcal{Z}(N, N'), \mathcal{O}), i = 1, 2$ are nontrivial for small values of N, N' .

Missing state phenomenon (see [2] Section 3) in the space of states is an unresolved problem for the original formulation of this $\beta\gamma$ system. Theorems 1.16, 1.18 and equation (4.122) indicate that our formalism is free from this drawback.

The problem of denominators The linear spaces $H^{i+\frac{\infty}{2}}$ are modules over polynomial algebra in coefficients of the Laurent series $\lambda(z) = \sum \lambda^k z^k, w_{ij}(z) = \sum w_{ij}^k z^k$ and $p_i(z) = \sum p_i^k z^k$. In particular, it is a module over $P = \mathbb{C}[\lambda^0, w_{ij}^0, p_i^0]$ which I identify with $\mathbb{C}[\lambda, w_{ij}, p_i]$. I can use these generators to define affine charts for $\text{Spec} P \setminus \{0\} = \dot{S}_+ \setminus \{0\}$:

$$\begin{aligned} U_0 &= \text{Spec}(\lambda)^{-1}P, \\ U_i &= \text{Spec}(p_i)^{-1}P, \\ U_{ij} &= \text{Spec}(w_{ij})^{-1}P. \end{aligned} \tag{1.23}$$

As usual, $(a_i)^{-1}R$ stands for localization of a ring R . By using the covering $\mathfrak{U} = \{U_0, U_i, U_{ij}\}$, I can define the Čech complex $\check{\text{Cech}}_e^\bullet(\mathfrak{U}, H^{i+\frac{\infty}{2}})$. The following problem arises in the theory of the b -ghost (see Section 2.4 in [6]). There is a map

$$r : H^{k+\frac{\infty}{2}} \rightarrow \check{\text{Cech}}^0(\mathfrak{U}, H^{k+\frac{\infty}{2}})$$

which is the diagonal localizations

$$H^{k+\frac{\infty}{2}} \rightarrow (\lambda)^{-1}H^{k+\frac{\infty}{2}} \oplus \bigoplus (p_i)^{-1}H^{k+\frac{\infty}{2}} \oplus \bigoplus (w_{ij})^{-1}H^{k+\frac{\infty}{2}}.$$

Fix k . The problem of denominators (of cochains in Čech complex) roughly is: Is it true that the map r defines an isomorphism of $H^{k+\frac{\infty}{2}}$ with $H^0(H^{k+\frac{\infty}{2}})$? And also is it true that $H^m(H^{k+\frac{\infty}{2}}) = 0$, for $m = 1, 2, 3, 4$? Here in the formulas, H^m stands for Čech cohomology of the sheaf associated with the P -module $H^{k+\frac{\infty}{2}}$. It is believed that the answer on this question is affirmative.

I prove a weaker version of this conjecture. The groups $H^m(H^{k+\frac{\infty}{2}})$ form the second page of some spectral sequence $H^m(H^{k+\frac{\infty}{2}}) \Rightarrow H^{k+m}C$ (Proposition 8.6). It turns out that the natural map $H^{k+\frac{\infty}{2}} \rightarrow H^kC$ is an isomorphism for $k = 0, \dots, 7$ (Proposition 8.5) and $H^kC = \{0\}, k = 4, \dots, 7$, which supports the conjecture. For the complete proof of the conjecture, it suffice to establish degeneration the spectral sequence in the second page.

The structure of the paper Results of this work rely heavily on the local structure of $\mathcal{Z}(N, N')$. The relevant analysis is done in Section 2, which is independent from the rest of the paper. There I prove the theorem (Proposition 2.35) that $\mathcal{Z}(N, N')$ is Gorenstein. Note that Gorenstein property is probably the only thing that is truly responsible for all the structures appearing in this paper.

In Section 3, I study what I call the Thom class th (3.5). This class will enable me to define "wrong direction" maps. My definition of $H^{i+\frac{\infty}{2}}$ as a limit of a bidirect system heavily relies on th . The section also contains some simple Tor functor construction for computation of local cohomology.

I devote Section 4 to systematic study of groups $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O})$. This includes verification of the nondegeneracy of the pairing (1.8) for $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O})$ and the proof of the functional equation (1.22). I introduce complexes $T_\bullet(\mathfrak{a})$, $S_\bullet(\mathfrak{a})$ and $\text{Fock}_\bullet^{\mathfrak{a}}[\delta, \delta']$ for computations of $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O})$.

I study N and N' dependencies in $H_{\mathcal{Z}(0, N')}^i(\mathcal{Z}(N, N'), \mathcal{O})$, which will be used for gluing these groups into $H^{i+\frac{\infty}{2}}$.

The limiting procedure used to define $H^{i+\frac{\infty}{2}}$ is rather delicate. Easy examples shows that nondegeneracy of the pairing between $H_{\mathfrak{a}}^i[\delta, \delta']$ $H_{\mathfrak{b}}^j[\delta, \delta']$ for all δ, δ' doesn't automatically imply it, as I mistakenly thought in the first draft of this paper, for $H^{i+\frac{\infty}{2}}$. Section 5 contains theorems which enable me to overcome this difficulty.

In Section 6, I finally introduce $H^{i+\frac{\infty}{2}}$ and prove Theorem 1.16. The definition of $H^{i+\frac{\infty}{2}}$ involves several limiting procedures. I verify that the result doesn't depend on the order of limits. I also give an elementary construction of $H^{3+\frac{\infty}{2}}$.

Section 7 is devoted to verification of Theorem 1.18.

I establish the denominator conjecture, discussed above, in Section 8.

Section 9 contains some speculations about the relation of this work to the earlier works on $\beta\gamma$ -systems [25],[25],[38].

The paper contains several appendices. For the reader's convenience, Appendix A contains basic facts about local cohomology, that are frequently used in the main text. Appendix B contains short review of the theory of Hibi algebras that is essential for Section 2. Some standard D -modules, that are used throughout the paper are introduced in Appendix C. I moved a number of Lemmas whose proofs are not very illuminating into Appendix E.

What was left out of scope To keep the manuscript within reasonable size limits, I put off discussion of some topics with a hope to return to them in future publications. Here are some of them.

The first topic is the structure of the vertex operator algebra on $H^{i+\frac{\infty}{2}}$ and the action of the affine

$\hat{\mathfrak{so}}_{10}$ and Virasoro algebra on it. The definition of the action is delicate: $\hat{\mathfrak{so}}_{10}$ doesn't act on $\mathcal{O}[\mathcal{Z}(N, N')]$ nor on $H^i_{\mathcal{Z}(0, N')}(\mathcal{Z}(N, N'), \mathcal{O})$. It emerges on $H^{i+\frac{\infty}{2}}$ in the limit.

The second topic is the Cousin complex. In representation theory Cousin complex appears under the name of the BGG resolution. Such BGG-like construction would certainly be relevant in my setup, since $\mathcal{Z}(N, N')$ is equipped with a suitable filtration. Even though the Kempf's algorithm [30] that defines the Cousin complex is compatible with my limiting procedure, the outcome of the algorithm resists at present to a simple characterization.

The third topic is the extension of the results to more general spaces. The Gorenstein property of $\mathcal{O}[\mathcal{Z}_{\mathcal{X}}(N, N')]$, $\mathcal{X} = \mathcal{C}$ is fundamental in all of my constructions. For cones over grassmannians, this property was established in [44]. For cones over isotropic symplectic grassmannians, it was proved in [41]. For cones over the spaces of full flags in simply-laced case, it was verified in [9]. In all these cases, the class **th** (1.19) is defined, which enables us to carry out many of the constructions from this paper. The question is how far one can push the theory. An issue arising on this way is discussed in Remark 4.130. It indicates that an upgrade of my method to a more systematic technique is desirable.

Fix a smooth Fano variety X with $\text{ind}(X) \geq 2$. Let $L^{-\otimes \text{ind}(X)} \cong K_X$, where K_X is the canonical line bundle over X . L can be used to define an affine cone \mathcal{X} over X . I wonder if the spaces $\mathcal{Z}_{\mathcal{X}}(N, N')$ are Gorenstein?

General conventions For an algebraic variety X over \mathbb{C} $\mathcal{O}[X]$ will stand for the algebra of regular algebraic functions on X .

The tensor product \otimes_R stands for the tensor product of R modules and \otimes for product of \mathbb{C} -vector spaces.

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2 The local structure of $\mathcal{Z}(N, N')$

The space $\mathcal{Z}(N, N')$ is singular. The goal of this section is to prove that its singularities are mild. This will be done by showing that $\mathcal{O}[\mathcal{Z}(N, N')]$ is Gorenstein, which will prove item 1 of Conjecture 1. I will establish this in Section 2.2 by degenerating $\mathcal{Z}(N, N')$ into a toric variety $\mathcal{H}(N, N')$ of a special kind. Here \mathcal{H} -stands for Hibi. He was the first who systematically studied algebras of homogeneous functions on such spaces. A Hibi algebra (see Appendix B for a brief review) has combinatorial description in terms

of a lattice (in the sense of Birkhoff). Gorenstein property of the algebra can be extracted directly from the underlying lattice. I do this for $\mathcal{O}[\mathcal{H}(N, N')]$ in Section 2.3. The lattice corresponding to $\mathcal{O}[\mathcal{H}(N, N')]$ is described in Section 2.1. In this section I continue to use notations from the Section 1.4.

2.1 Basic definitions

Spinors and the cone \mathcal{C} It will be useful to have a more invariant description of the cone \mathcal{C} (1.11). The set

$$E = \{(0), (ij), (k) | 1 \leq i < j \leq 5, 1 \leq k \leq 5\}$$

labels a weight basis for an irreducible spinor representation \mathring{S}_+ of the complex group $\text{Spin}(10)$ with respect to the maximal torus \mathbf{T}^5 . Let \mathring{V} the fundamental representations of the complex $\text{SO}(10)$ with a \mathbf{T}^5 -weight basis labelled by the set of symbols

$$G = \{1, \dots, 5, 1^*, \dots, 5^*\}.$$

See [36] for details. The direct sum of \mathring{S}_+ with its dual \mathring{S}_- comprise an irreducible representation of the Clifford algebra $Cl = Cl(\mathring{V}, (\cdot, \cdot))$: $Cl \otimes (\mathring{S}_+ + \mathring{S}_-) \rightarrow \mathring{S}_+ + \mathring{S}_-$. Restriction to $\mathring{V} \subset Cl$ defines maps $\mathring{V} \otimes \mathring{S}_+ \rightarrow \mathring{S}_-$, $\mathring{V} \otimes \mathring{S}_- \rightarrow \mathring{S}_+$. Their adjoints are the Γ -maps

$$\text{Sym}^2(\mathring{S}_+) \rightarrow \mathring{V}, \quad \text{Sym}^2(\mathring{S}_-) \rightarrow \mathring{V}.$$

In \mathbf{T}^5 -weight bases $\{\theta_\beta, \beta \in E\}$ for \mathring{S}_+ and $\{v_s, s \in G\}$ for \mathring{V} , the Γ -matrices $\Gamma_{\alpha\beta}^s$ are defined by the formula $\Gamma(\theta_\alpha, \theta_\beta) = \Gamma_{\alpha\beta}^s v_s$. Summation over repeated indices will always be assumed. Let us introduce a more uniform notations for coordinates on the space of spinors: $\theta = \lambda^{(0)}\theta_0 + \lambda^{(ij)}\theta_{ij} + \lambda^{(i)}\theta_i \in \mathring{S}_+$. Equations

$$\Gamma_{\alpha\beta}^s \lambda^\alpha \lambda^\beta = 0, s \in G, \lambda^\beta \theta_\beta \in \mathring{S}_+ \quad (2.1)$$

after the identification

$$\lambda = \lambda^{(0)}, w_{ij} = \lambda^{(ij)}, p_i = \lambda^{(i)}$$

coincide with (1.13). More invariantly than in (1.14), I define \mathring{A} as

$$\mathring{A} = \mathbb{C}[\lambda^\beta] / (\Gamma_{\alpha\beta}^s \lambda^\alpha \lambda^\beta).$$

In this description, the $\mathbb{C}^\times \times \text{Spin}(10)$ -symmetries of the cone \mathcal{C} becomes manifest. The factor \mathbb{C}^\times stands for dilations of the cone.

Defining equations for $\mathcal{Z}(N, N')$ The scheme $\mathcal{Z}(N, N')$ (1.15) is an affine algebraic manifold. Occasionally, following [44], I will call it *Quantum Isotropic Grassmannian*. The algebra of polynomial functions $A_N^{N'} = \mathcal{O}[\mathcal{Z}(N, N')]$ ([36]) is generated by the variables

$$\{\lambda^{\beta^l} | N \leq l \leq N', \beta \in E\}$$

where β^l is a multi-index. Defining relations of $A_N^{N'}$ are

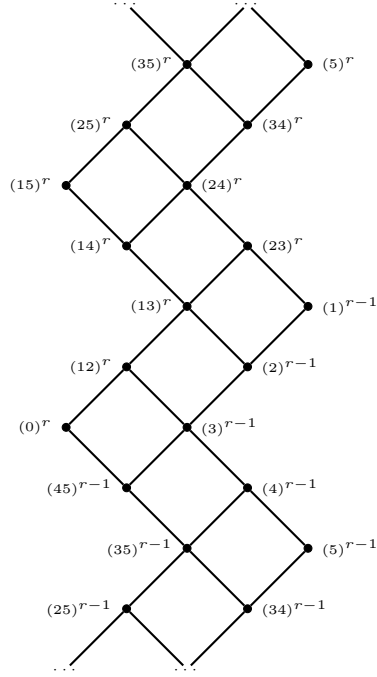
$$\Gamma^{s^k} = \sum_{l+l'=k} \Gamma_{\alpha\beta}^s \lambda^{\alpha^l} \lambda^{\beta^{l'}}, \quad N \leq l, l' \leq N', 2N \leq k \leq 2N' \quad s \in G, \quad (2.2)$$

$$\sum_{2N \leq l \leq 2N'} \Gamma^{s^k} z^k = \Gamma_{\alpha\beta}^s \lambda^\alpha(z) \lambda^\beta(z), \quad \lambda^\beta(z) = \sum_{N \leq l \leq N'} \lambda^{\beta^l} z^l. \quad (2.3)$$

Coordinates λ^{β^k} (2.3) are the decorated Fourier coefficients x^{i^k} (1.6)

Remark 2.4 *From this definition it is obvious that $\mathbb{C}^\times \times \text{Spin}(10)$ is a groups of symmetries of the algebra $A_N^{N'}$.*

The fundamental Hasse diagram The following diagram \hat{E} is fundamental for Quantum Isotropic Grassmannians:



(2.5)

Its vertices are decorated elements of the set E . I embed $E \rightarrow \hat{E}$, $\beta \rightarrow \beta^0$ as a full sub-diagram. I will denote elements $\beta^r \in \hat{E}$ by α . Then

$$u(\alpha) := r \text{ for } \alpha = \beta^r. \quad (2.6)$$

It will be convenient to work with a slightly more general family of algebras than $A_N^{N'}$. The Hasse graph \hat{E} defines a *poset* (partially ordered set) structure on the set of its vertices (see Appendix B for details). Suppose $L \subset \hat{E}$. I set

$$\mathbb{C}[L] = \mathbb{C}[\lambda^\alpha], \quad \Lambda[L] = \Lambda[\xi^\alpha], \quad \alpha \in L.$$

Fix $K \subset L$.

$$i(K) = \left\{ \sum_{\alpha \in K} \lambda^\alpha \mathbb{C}[L] \right\} \subset \mathbb{C}[L]$$

is an ideal of $\mathbb{C}[L]$.

Algebraic variety $Z(\delta, \delta')$ A *segment* or an *interval* $[\delta, \delta']$ is a subset $\{\alpha \in \hat{E} | \delta \leq \alpha \leq \delta'\}$. The set of generators of $A_N^{N'}$ is $\{\lambda^\alpha | \alpha \in [(0)^N, (1)^{N'}]\}$.

Occasionally I will denote \hat{E} by $[-\infty, \infty]$.

Definition 2.7 Fix $X \subset [(0)^N, (1)^{N'}]$ for some N, N' . The algebra $A[X]$ is a quotient of $\mathbb{C}[(0)^N, (1)^{N'}]$. The ideal of relations $\mathfrak{r}[X]$ is generated by $\Gamma^{\hat{s}}$ (2.2) and $\{\lambda^\alpha | \alpha \notin X\}$. Assignment $X \Rightarrow A[X]$ is a contravariant functor from the category of finite subsets of \hat{E} (morphisms are inclusions) to the category of algebras. When X is the interval $[\delta, \delta']$, the algebra $A[X]$ will be denoted by $A[\delta, \delta']$. By definition $A_N^{N'} := A[(0)^N, (1)^{N'}]$. The algebra $A[\delta, \delta']$ was denoted by $A_\delta^{\delta'}$ in [36]. Let K be a subset of X . By abuse of notations, I will denote the ideal $\{\sum_{\alpha \in K} \lambda^\alpha A[X]\} \subset A[X]$ by $\mathfrak{i}[K]$.

I define

$$\begin{aligned} Z[\delta, \delta'] &:= \text{Spec } A[\delta, \delta'] \\ S_+[\delta, \delta'] &:= \text{span } \langle \theta_{\beta^k} = \theta_\beta z^k | \beta^k \in [\delta, \delta'] \rangle \end{aligned} \quad (2.8)$$

The shift operator τ

$$\tau(\beta^r) = \beta^{r+1} \quad \beta^r \in \hat{E} \quad (2.9)$$

is an automorphism of the graded lattice \hat{E} (see Definition B.1 and Example B.2). Note that the map of the lattice (2.9) define homomorphism:

$$\tau(\lambda^{\beta^n}) = \lambda^{\beta^{n+1}} \text{ induces an isomorphism } \tau : A[\delta, \delta'] \rightarrow A[\tau(\delta), \tau(\delta')]. \quad (2.10)$$

The algebra $A[-\infty, \infty] := \varprojlim_{\delta, \delta'} A[\delta, \delta']$ admits an involution σ . This involutions defines an anti-involution of the poset

$$\sigma : \hat{E} \rightarrow \hat{E}. \quad (2.11)$$

Geometrically σ is a central symmetry about a center O which is a bisector of the segment $|(25)^0, (34)^0|$

(2.5) The map σ is induced by isomorphisms

$$\sigma : A[\delta, \delta'] \rightarrow A[\sigma(\delta'), \sigma(\delta)]. \quad (2.12)$$

(see Appendix A in [36]).

Convention 1 We will encounter many constructions that depend on the interval from \hat{E} , e.g $A[\delta, \delta']$, $S_+[\delta, \delta']$, $\mathbb{C}[\delta, \delta']$. To avoid clutter in notations, I will often drop dependence on the interval in the case when the interval can be extracted from the context. In this case $\mathbb{C}[\delta, \delta']$ will be denoted by P

Algebras with the straightening law

Definition 2.13 Let A, P be the algebras based on $[\delta, \delta']$. A monomial $\lambda^{\alpha_1} \dots \lambda^{\alpha_n}$ in P is said to be standard if $\delta' \leq \alpha_1 \leq \dots \leq \alpha_n \leq \delta$ with respect to the partial order in $[\delta, \delta'] \subset \hat{E}$. An element $x \in A$ is said to be a standard monomial if one of its pre-images in P is standard.

Proposition 2.14 (See [36]) Fix $[\delta, \delta']$ on which algebra A is based.

- i. Each of the defining equations of A contains a unique monomial $\lambda^\alpha \lambda^\beta$, which is called a clutter, such that α and β are not comparable.
- ii. More precisely equations have the form

$$\lambda^\alpha \lambda^\beta \pm \lambda^{\alpha \vee \beta} \lambda^{\alpha \wedge \beta} - \sum_{\gamma < \alpha \wedge \beta, \delta > \alpha \vee \beta} \pm \lambda^\gamma \lambda^\delta = 0 \quad (2.15)$$

where $\lambda^\alpha \lambda^\beta$ is a clutter.

- iii. Standard monomials define a basis for A .
- iv. A is a Cohen-Macaulay algebra.

Definition 2.16 An algebra with a set of generators labeled by the elements of a lattice F with relations (2.15) is said to be an algebra with *straightening laws* (see e.g. [20]) if standard monomials form a basis. Thus $A[\delta, \delta']$ and in particular $A_N^{N'}$ are algebras with straightening laws.

I will occasionally use algebras based on semi-intervals:

$$A(\delta, \delta'] := A[\delta, \delta'] / (\lambda^\delta), \quad A[\delta, \delta') := A[\delta, \delta'] / (\lambda^{\delta'}). \quad (2.17)$$

Proposition 2.18 The standard monomials defined in terms of the order on semi-interval form a basis for A based on $(\delta, \delta']$ or $[\delta, \delta')$.

Proof. Let us work out the case of $(\delta, \delta']$. Elements δ, δ' are comparable with all the elements of $[\delta, \delta']$. Thus the Gröbner basis for the defining ideal $\mathfrak{c} = \mathfrak{r}[\delta, \delta'] + (\lambda^\delta)$ such that $P/\mathfrak{c} \cong A$ besides (2.15) contains only λ^δ (see more on application of Gröbner basis technique to in [36]). I conclude from this that the standard monomials defined in terms of the order on $(\delta, \delta']$ form a basis for A .

The arguments for $[\delta, \delta')$ are similar. ■

2.2 The toric degeneration of $Z[\delta, \delta']$

The algebra of functions on the toric degeneration announced in the title of this section will be constructed combinatorially as a contraction of the algebra A based on $[\delta, \delta']$. More precisely, I will define an algebra A_h which is isomorphic to $A \otimes \mathbb{C}[[h]]$ as $\mathbb{C}[[h]]$ -module. I will define

$$Hibi[\delta, \delta'] := A_h[\delta, \delta']/(h).$$

The formal family A_h has the property that

$$A_h \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h)) \cong A \otimes \mathbb{C}((h)).$$

In this section, I will use some notation of Appendix B.

For construction of A_h , I use some ideas from [44]. They define a similar deformation in the context of ordinary Grassmannians $\text{Gr}(n, m+n)$. As in my case, their algebra is controlled by a Hasse diagram. Sottile and Sturmfels's idea was to modify defining relations of the algebra by using a special function associated with the diagram.

The function l To define $l : \hat{E} \rightarrow \mathbb{Z}_{>0}$, I embed the graph \hat{E} into $\mathbb{Z}^2 \subset \mathbb{R}^2$. For this, I use the straightforward identification of the physical surface (e.g. a computer screen or a sheet of a paper) on which the diagram 2.5 is drawn with the plane \mathbb{R}^2 . To be more precise, I can characterize this embedding by a map

$$f : \hat{E} \rightarrow \mathbb{R}^2. \tag{2.19}$$

To describe it, I fix a sublattice $S \subset \mathbb{Z}^2$ generated by vectors $u = (2, 2), v = (-2, 2)$. I characterize f by the set of conditions:

1. $f(\alpha) \in (0, -1) + S$.
2. If $\alpha > \beta$, then $f(\alpha) - f(\beta) \in \{u, v\}$;
3. $f((34)^0) = (2, 0), f((35)^0) = (0, 2), f((25)^0) = (-2, 0), f((24)^0) = (0, -2)$.

$$l(\alpha) = \|f(\alpha)\|^2 \text{ where } \|(x, y)\|^2 = x^2 + y^2.$$

For finite subset $N \subset \hat{E}$ define

$$\text{Core}(N) := \{\alpha \in N \mid \exists \beta \neq \alpha, \rho(\alpha) = \rho(\beta)\}. \tag{2.20}$$

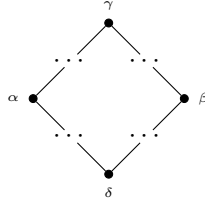
The function $\rho = \rho_{\hat{E}}$ is defined in (B.3) *Capacity* of N is

$$\text{Cap}N := |\text{Core}(N)|. \quad (2.21)$$

The next proposition underlies many results of this paper.

Proposition 2.22 1. $\forall \alpha \in \hat{E} \, l(\alpha) > 0$

2. Pick any rectangle in the diagram (2.5):



$$(2.23)$$

Then

$$l(\alpha) + l(\beta) = l(\gamma) + l(\delta). \quad (2.24)$$

3. Pick α, β as in (2.23). Suppose $\exists t \in \mathbb{Z}^2, \gamma', \delta' \in \hat{E}$ such that

$$f(\gamma') = (f(\alpha) + f(\beta))/2 + t, f(\delta') = (f(\alpha) + f(\beta))/2 - t.$$

If

$$l(\alpha) + l(\beta) \geq l(\gamma') + l(\delta'), \quad (2.25)$$

then

$$\gamma', \delta' \in [\alpha \wedge \beta, \alpha \vee \beta]. \quad (2.26)$$

Moreover, equality holds iff γ', δ' coincides with opposite corners of the rectangle $[\alpha \wedge \beta, \alpha \vee \beta]$.

Proof.

1. $l(\alpha) > 0$ because $f^{-1}(0) = \emptyset$.

2. Set $g = f(\delta)$ $a = f(\alpha) - f(\delta), b = f(\beta) - f(\delta)$. By construction $f(\gamma) - f(\delta) = a + b$. Equation (2.24) is equivalent to

$$\|g\|^2 + \|g + a + b\|^2 - \|g + a\|^2 - \|g + b\|^2 = 2(a, b) = 0.$$

The equality holds true because vectors a and b are proportional to u and v (2) which are orthogonal by construction.

3. Suppose (α, β) is a clutter. The quantity $\|(\mathbf{f}(\alpha) - \mathbf{f}(\beta))/2\|^2 = Q(c)$ depends only on the capacity $c = \text{Cap}[\alpha \wedge \beta, \alpha \vee \beta]$ (2.21). Under my assumptions capacity ranges from one to three and

$$Q(1) = 4, Q(2) = 10, Q(3) = 20.$$

The center of symmetry of $\mathbf{f}([\alpha \wedge \beta, \alpha \vee \beta])$ is located at $(\mathbf{f}(\alpha) + \mathbf{f}(\beta))/2$. The square of the radius $R^2(c)$ of the circumscribed circle $\text{Circle}[\alpha \wedge \beta, \alpha \vee \beta]$ about $\mathbf{f}([\alpha \wedge \beta, \alpha \vee \beta])$ as a function of capacity c is equal to

$$R(1) = 4, R(2) = 10, R(3) = 20.$$

It implies that that $Q(c) = R^2(c)$. I set $\mathbf{f}(\alpha) = m$, $\mathbf{f}(\beta) = n$. From the identity

$$\begin{aligned} & \| (m+n)/2 + t \|^2 + \| (m+n)/2 - t \|^2 - \|m\|^2 - \|n\|^2 = \\ & 2(\|t\|^2 - \|(m-n)/2\|^2) \end{aligned}$$

I infer that if the inequality (2.25) holds, then the vector $(m+n)/2+t$ lies inside $\text{Circle}[\alpha \wedge \beta, \alpha \vee \beta]$ or on its boundary. Then (2.26) holds. When the inequality becomes equality, the vector $(m+n)/2+t$ lies on the circle $\text{Circle}[\alpha \wedge \beta, \alpha \vee \beta]$ and γ', δ' the pair of opposite corners of $[\alpha \wedge \beta, \alpha \vee \beta]$.

■

Remark 2.27 Algebra $A[\delta, \delta']$ has a symmetry group containing Aut (1.20) The action of $g = (t, q, z) = (t, q, z_1, \dots, z_5) \in \text{Aut}$ is

$$\Pi(g)\lambda^\alpha = \hat{v}_\alpha(t, q, z)\lambda^\alpha = tq^{u(\alpha)}v_\beta(z)\lambda^\alpha,$$

or in more details:

$$\Pi(g)\lambda^{(0)^t} = tq^l \det^{-\frac{1}{2}}(z)\lambda^{(0)^t}, \tag{2.28}$$

$$\Pi(g)\lambda^{(ij)^t} = tq^l \det^{-\frac{1}{2}}(z)z_i z_j \lambda^{(ij)^t},$$

$$\Pi(g)\lambda^{(k)^t} = tq^l \det^{\frac{1}{2}}(z)z_k^{-1} \lambda^{(k)^t}.$$

See (2.6) for $u(\alpha)$.

The following proposition gives an additional insight into the structure of the defining relations of A .

Proposition 2.29 *Let A be based on $[\delta, \delta']$. For any clutter $\alpha, \beta \in [\delta, \delta'] \subset \hat{E}$, there is a unique defining relation (2.15) of A that can be written in the form*

$$\lambda^\alpha \lambda^\beta \pm \lambda^{\alpha \vee \beta} \lambda^{\alpha \wedge \beta} = \sum_{\mathbf{f}(\gamma) = (\mathbf{f}(\alpha) + \mathbf{f}(\beta))/2 + t, \mathbf{f}(\delta) = \mathbf{f}(\alpha) + \mathbf{f}(\beta) - t} \pm \lambda^\gamma \lambda^\delta$$

with \mathbf{f} as in (2.19). The summation is taken over $t \in \mathbb{Z}^2$ such that the corresponding γ', δ' satisfy

$$\mathbf{l}(\alpha) + \mathbf{l}(\beta) < \mathbf{l}(\gamma') + \mathbf{l}(\delta'). \quad (2.30)$$

Proof.

$\mathbf{T} \times \mathbf{T}^5 \subset \mathbf{T} \times \mathrm{SO}(10)$ contains two one-parametric subgroups

$$x : \mathbb{C}^\times \rightarrow \mathbf{T} \times \mathbf{T}^5 \quad x(z) = (1, 1, z^2, z^4, z^2, 1),$$

$$y : \mathbb{C}^\times \rightarrow \mathbf{T} \times \mathbf{T}^5 \quad y(w) = (w^{-16}, 1, w^2, w^4, w^6, w^8).$$

One-parametric subgroups $x(z)$ and $y(w)$ designed in a such a way that

$$\Pi(x(z))\lambda^\alpha = z^{\mathbf{f}_1(\alpha)}\lambda^\alpha, \Pi(y(w))\lambda^\alpha = w^{\mathbf{f}_2(\alpha)}\lambda^\alpha.$$

$\mathbf{f}_1(\alpha), \mathbf{f}_2(\alpha)$ are components of the map $\mathbf{f}(\alpha)$. Defining relations $\Gamma^{\mathbf{s}^k}$ (2.2) are $\mathbf{T} \times \tilde{\mathbf{T}}^5$ -weight vectors. Thus all monomials scale by the same factor under $\Pi(x(z)y(w))$ -action. The weight of the clutter $\lambda^\alpha \lambda^\beta$ is $z^{\mathbf{f}_1(\alpha) + \mathbf{f}_1(\beta)} w^{\mathbf{f}_2(\alpha) + \mathbf{f}_2(\beta)}$. So $\mathbf{f}(\alpha) + \mathbf{f}(\beta) = \mathbf{f}(\gamma') + \mathbf{f}(\delta')$. Let t be such that

$$\mathbf{f}(\gamma') = (\mathbf{f}(\alpha) + \mathbf{f}(\beta))/2 + t,$$

$$\mathbf{f}(\delta') = (\mathbf{f}(\alpha) + \mathbf{f}(\beta))/2 - t.$$

I know that $\Gamma^{\mathbf{s}^k}$ contains a unique clutter $\lambda^\alpha \lambda^\beta$. It corresponds to a rectangle $[\alpha \wedge \beta, \alpha \vee \beta]$ in the diagram (2.5). By Proposition 2.22, the inequality (2.30) is violated precisely at $\gamma', \delta' \in [\alpha \wedge \beta, \alpha \vee \beta]$. Straitedened relation forbid such γ', δ' . ■

The family A_h Let A be based on $A[\delta, \delta']$.

I use the function \mathbf{l} to define the flat family of algebras A_h . It is a quotient of $\mathbb{C}[[h]]$ by the ideal \mathfrak{r} generated by

$$\lambda^\alpha \lambda^\beta \pm \lambda^{\alpha \vee \beta} \lambda^{\alpha \wedge \beta} = \sum_{\mathbf{f}(\gamma) = \mathbf{f}(\alpha) + \mathbf{f}(\beta)/2 + t, \mathbf{f}(\delta) = \mathbf{f}(\alpha) + \mathbf{f}(\beta) - t} \pm h^{\mathbf{l}(\gamma) + \mathbf{l}(\delta) - \mathbf{l}(\alpha) - \mathbf{l}(\beta)} \lambda^\gamma \lambda^\delta. \quad (2.31)$$

It is worth pointing that by Proposition 2.22, the exponents of h in the relation are positive. Relations (2.31) define a straightening law on A_h , whose proof is the same as in the case when $h = 1$ (see Proposition 2.14). This means that standard monomials define a $\mathbb{C}[[h]]$ -basis for A_h . As a result, A_h is flat over $\mathbb{C}[[h]]$. After reduction $\mod h$, relations (2.31) become

$$\lambda^\alpha \lambda^\beta \pm \lambda^{\alpha \vee \beta} \lambda^{\alpha \wedge \beta} = 0. \quad (2.32)$$

These equations define a toric variety announced in the title of the section. The algebra $Hibi[\delta, \delta']$ is the quotient of P by (2.32). Introduce a notation:

$$\mathcal{H}[\delta, \delta'] := \text{Spec } Hibi[\delta, \delta'].$$

As the notation suggests, $Hibi$ is an example of a Hibi algebra. These algebras were studied in [27]. See also Appendix B for background information on Hibi algebras.

Triviality of the deformation A_h on $\text{Spec } \mathbb{C}((h))$ is verified in the next proposition.

Proposition 2.33 *There is an isomorphism*

$$u : A \otimes_{\mathbb{C}} \mathbb{C}((h)) \rightarrow A_h \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h)).$$

Proof. The isomorphism is defined by the formula $u(\lambda^\alpha) = h^{l(\alpha)} \lambda^\alpha$. ■

Remark 2.34 *The substitution $\lambda^{(0)^n} \rightarrow \sqrt{-1} \lambda^{(0)^n}$, $\lambda^{(j)^n} \rightarrow \sqrt{-1} \lambda^{(j)^n}$, $j = 1, \dots, 5$ transforms relations (2.32) to the Hibi form (B.8). To see this, the reader should look at the defining relations for A where terms (2.32) are written explicitly as in [36] (Eqs. 56).*

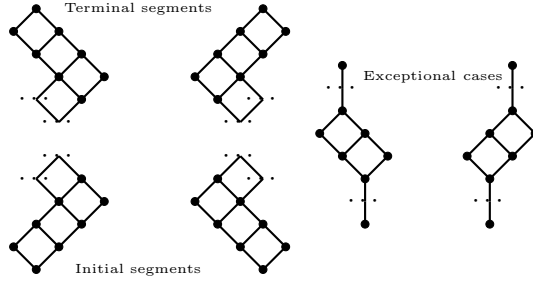
2.3 Classification of $Z[\delta, \delta']$ with Gorenstein singularities

Classification of Gorenstein $Z[\delta, \delta']$ can be formulated in terms of the geometry of ends of the interval $[\delta, \delta'] \subset \hat{\mathbb{E}}$.

Proposition 2.35 *The algebra A based on $[\delta, \delta']$ is Gorenstein iff the neighborhoods of the ends of the Hasse diagram of $[\delta, \delta']$ don't look like those on the following picture.*

Commutative algebra background Before giving the proof of the proposition, I need to remind the reader some definition from commutative algebra (see [11] and [34] for details).

I will be using a definition of a Cohen-Macaulay and Gorenstein algebra that I borrowed from [45]. It is designed for the graded algebras. It has an advantage that it makes construction of a dualizing module more explicit.



Definition 2.36

$$R = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{r} \quad (2.37)$$

is a graded Cohen-Macaulay algebra of dimension \dim iff the minimal graded free $P = \mathbb{C}[x_1, \dots, x_n]$ -resolution

$$0 \leftarrow R \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \leftarrow \dots \leftarrow F_{n-d-1} \xleftarrow{d_{n-d}} F_{n-\dim} \leftarrow 0$$

satisfies $H^i(\text{Hom}_P(F_\bullet, P)) = \{0\}, i \neq n - \dim$. The P -module $\omega_A = H^{n-\dim}(\text{Hom}_P(F_\bullet, P))$ is in fact an A -module. It is called the canonical module associated with the presentation (2.37). F_\bullet is self-dual iff R is Gorenstein.

The reader may wish to check with ([43] and [4] Proposition 5.1) for the proof of the equivalence to the standard definition ([11] and [34]).

There is yet another characterization of Gorenstein algebras due to Stanley. Let $V = \bigoplus_{i \geq i_0} V_i$. The Poincaré series $V(t)$ is the formal function $\sum_{i \geq i_0} \dim V_i t^i$. If R is a graded algebra $R = \bigoplus_{i \geq 0} R_i$, then $R(t)$ is the Poincaré series of the underlying vector space.

Proposition 2.38 [45] *Let R be a graded algebra. Suppose R is a Cohen-Macaulay integral domain of Krull dimension $d = \dim R$. Then R is Gorenstein iff for some $p \in \mathbb{Z}$,*

$$R(1/t) = (-1)^d t^p R(t). \quad (2.39)$$

The proof of Proposition 2.35 and more To determine under what conditions on δ , and δ' the algebra A based on $[\delta, \delta']$ is Gorenstein, I will use the toric degeneration from Section 2.2. I will reduce the problem to the similar problem about Hibi algebra *Hibi*. For algebras based on distributive lattices (this applies to us) the problem was completely solved in [27]. The answer can be read off from the Hasse diagram $[\delta, \delta']$. By Remark B.9, I have to analyze maximal chains (see Appendix B) in the join-irreducible subposet $B[\delta, \delta']$ of $[\delta, \delta']$ (the letter B is after Birkhoff).

A poset X is *equidimensional* if its maximal chains have equal length. A lattice X is *Gorenstein* if in the subposet $B(X) \subset X$ of join-irreducible elements is equidimensional.

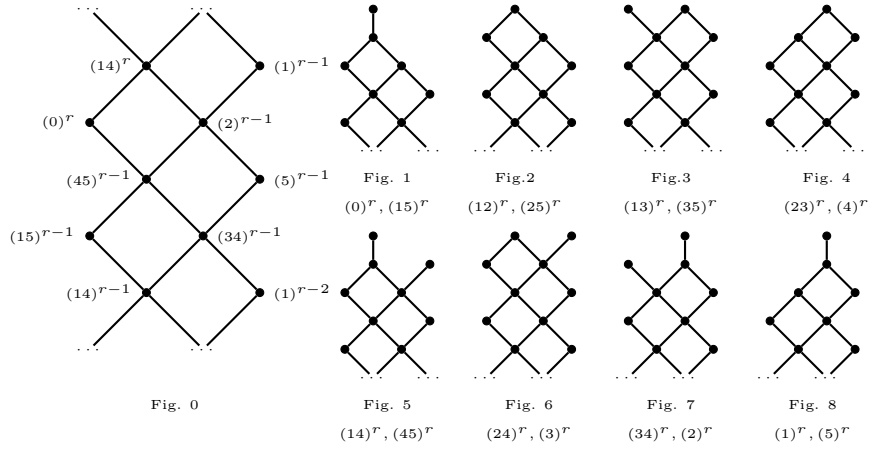
The action τ (2.9) on \hat{E} induces a periodic structure on $B[-\infty, \infty]$. $B[-\infty, \infty]$ is a graded poset (see Example B.4). With the help of τ -periodicity and direct inspection I verify that $\rho_B^{-1}([a, b]) \subset B[-\infty, \infty]$ is an equidimensional poset (see Fig 0 for $\rho_B^{-1}([-3, 1])$). Suppose that a and b are such that

$$\emptyset \neq \rho_B^{-1}([a, b]) \subset B[\delta, \delta']$$

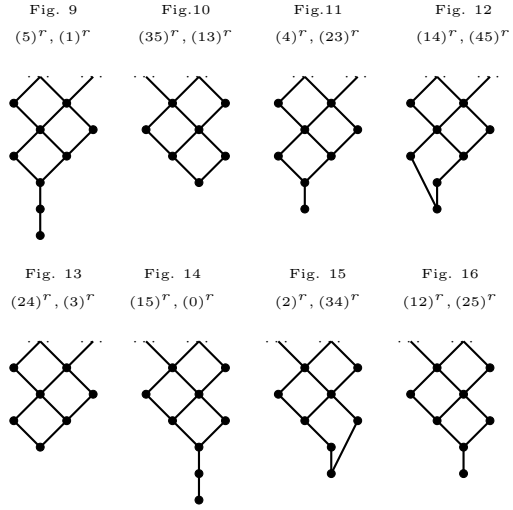
is maximal. The geometry of \hat{E} dictates that $B[\delta, \delta'] \setminus \rho_B^{-1}([a, b])$ is a union $B(\delta) \cup B(\delta')$ of two posets-*the ends*. They satisfy $B(\delta) < B(\delta')$.

If $B[\delta, \delta']$ is not Gorenstein, then at least one of the ends $B(\delta)$ or $B(\delta')$ fails to be an equidimensional poset.

The structure of $B(\delta')$, the terminal part, depends only on δ' . The list of Hasse diagrams $B(\delta')$ as functions of δ' is given on Fig. 1-8.



In complete analogy with the terminal part the list of Hasse diagrams $B(\delta)$ of initial parts as a function of δ is given on Fig. 9-16.



Note that $B[\delta, \delta']$ is non-Gorenstein iff it is having ends as in Fig.5 or 7 or 12 or 15.

I can define the ends $U_{left}(\delta)$ and $U_{right}(\delta')$ of $[\delta, \delta']$ the same way as for $B[\delta, \delta']$. Indeed, \hat{E} is a graded poset (see Example B.2). Suppose that a and b are such that

$$\emptyset \neq \rho_{\hat{E}}^{-1}([a, b]) \subset [\delta, \delta']$$

is maximal. Then $[\delta, \delta'] \setminus \rho_{\hat{E}}^{-1}([a, b])$ is a union $U_{left}(\delta) \cup U_{right}(\delta')$ of the ends which satisfy $U_{left}(\delta) < U_{right}(\delta')$.

It is a matter of finite check to see that $U_{left}(\delta)$ and $U_{right}(\delta')$ completely determine $B(\delta)$ and $B(\delta')$ respectively. $U_{right}(\delta')$ with δ' as on Fig. 5 and 7 are precisely the terminal segments on the diagram (2.35). Similarly, $U_{left}(\delta)$ with δ as on Fig. 12 and 15 are precisely the initial segments on the diagram (2.35). ■

Introduce a pair of totally ordered subsets of \hat{E} :

$$\begin{aligned} M_1^+ &= \{\dots < (35)^{r-1} < (4)^{r-1} < (3)^{r-1} < (12)^r < (13)^r < (23)^r < (24)^r < (25)^r < (35)^r < \dots \quad r \in \mathbb{Z}\} \\ M_1^- &= \{\dots < (24)^r < (34)^r < (35)^r < (45)^r < (3)^r < (2)^r < (13)^{r+1} < (14)^{r+1} < (24)^{r+1} < \dots \quad r \in \mathbb{Z}\} \end{aligned} \quad (2.40)$$

For definition of M_i^\pm see Lemma E.3. I can sum up the above discussion in the following theorem.

Theorem 2.41 *The interval $[\delta, \delta']$ is Gorenstein iff capacity Cap (2.21) satisfies*

$$\begin{aligned} &\text{Cap}[\delta, \delta'] = 0 \text{ or } \text{Cap}[\delta, \delta'] = 1, \text{ or} \\ &\text{Cap}[\delta, \delta'] \geq 3 \text{ and the end points satisfy :} \\ &\delta \in \hat{E} \setminus M_2^+ = M_1^+ \sqcup M_3^+, \\ &M_2^+ = \{(14)^r, (45)^r, (34)^r, (2)^r | r \in \mathbb{Z}\} \text{ and} \\ &\delta' \in \hat{E} \setminus M_2^- = M_1^- \sqcup M_3^-, \\ &M_2^- = \{(12)^r, (25)^r, (23)^r, (4)^r | r \in \mathbb{Z}\}. \end{aligned} \quad (2.42)$$

The length (B.5) of the maximal chain in $[\delta, \delta']$ in terminology of Appendix B is the rank

$$\text{rk}(\delta, \delta') \text{ of } [\delta, \delta'].$$

I deduce from Remark B.9 a pair of corollaries.

Corollary 2.43 *The algebra $\text{Hibi}[\delta, \delta']$ is Gorenstein iff δ, δ' satisfy (2.42).*

I leave verification to the reader that ρ (B.3) satisfies

$$\rho(\delta') - \rho(\delta) = \text{rk}(\delta, \delta'), \quad (2.44)$$

$$\rho\sigma(\alpha) = -\rho(\alpha) + 10 \quad (2.45)$$

where σ as in (2.11).

Corollary 2.46 $\dim \operatorname{Spec} \operatorname{Hibi}[\delta, \delta'] = \operatorname{rk}(\delta, \delta') + 1$.

The next proposition is the central result of this section.

Proposition 2.47 1. The algebra $A[\delta, \delta']$ is Gorenstein iff δ, δ' satisfy (2.42).

2. $\dim A[\delta, \delta'] = \operatorname{rk}(\delta, \delta') + 1$.

Proof. By Corollary 4.2 [20] A is a Cohen-Macaulay algebra. This implies that $A_h \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ has the Cohen-Macaulay property. From the line of equalities

$$\dim_{\mathbb{C}} A_i = \dim_{\mathbb{C}((h))} A_{hi} \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h)) = \operatorname{rk}_{\mathbb{C}[[h]]} A_{hi} = \dim_{\mathbb{C}} \operatorname{Hibi}_i,$$

I conclude that $A(t) = \operatorname{Hibi}(t)$.

Lemma 2.48 Algebra A based on any interval $[\delta, \delta']$ is an integral domain.

Proof. The algebra $\operatorname{Hibi} = A_h/(h)$ is an integral domain [27]. Since A_h is $\mathbb{C}[[h]]$ -flat, A_h is an integral domain. Thus $A_h \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h)) \cong A \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ and A are integral domains. ■

By Corollary 2.43, Hibi is Gorenstein. By Proposition 2.38, $\operatorname{Hibi}(t)$ satisfies (2.39). Thus $A(t)$ satisfies the same equation. By the same proposition A is Gorenstein. The dimension formula follows from the comparison of degrees of Hilbert polynomials of Hibi and A ■

Example 2.49 1. Suppose $N, N' \geq 0$. From Corollary 2.47 follows that $\dim_{K_{rull}} A_N^{N'} = 11 + 8(N' - N)$.

2. The algebra $A_N^{N'} = A[(0)^N, (1)^{N'}]$ is Gorenstein $\forall N < N'$. This verifies Conjecture 1 item 1 for $\mathcal{X} = \mathcal{C}$.

Filtration by Gorenstein sub-schemes Maps between local cohomology $H_{X \cap Z}^i(X, \mathcal{O})$ and $H_Z^i(Y, \mathcal{O})$ of a pair of schemes $X \subset Y$ often become more tractable if $\operatorname{codim} X = 1$. For a pair $Z[\delta', \beta] \supset Z[\delta, \beta]$ of Gorenstein schemes corresponding to a pair of intervals $[\delta', \beta] \supset [\delta, \beta]$, I will now construct a sequence of Gorenstein schemes Z_i

$$Z[\delta', \beta] = Z_0 \supset Z_1 \supset \cdots \supset Z_n = Z[\delta, \beta]$$

such that $\dim Z_{i-1} - \dim Z_i = 1$.

Proposition 2.50 *Let $[\delta, \beta] \supset [\delta', \beta]$ be intervals satisfying (2.42) such that $\delta' \in M_1^+$. Then there is a sequence of intervals*

$$[\delta, \beta] = [\delta_1, \beta] \supset [\delta_2, \beta] \supset \cdots \supset [\delta_n, \beta] = [\delta', \beta] \quad (2.51)$$

$\delta_i \in M_1^+, i > 1$ such that they satisfy (2.42) and

$$\delta_i < \delta_{i+1}, \text{ and in particular } \rho(\delta_{i+1}) = \rho(\delta_i) + 1, \quad (2.52)$$

and $\delta_1 = \delta, \delta_n = \delta'$.

Proof. Suppose that the capacities (2.21) satisfy $\text{Cap}[\delta, \beta] \geq 3, \text{Cap}[\delta', \beta] \geq 3$.

By the assumption $\delta' \in M_1^+$. Conditions (2.42) imply that $\delta \notin M_2^+ \Rightarrow \delta \in \hat{E} \setminus M_2^+ = M_1^+ \sqcup M_3^+ = M_1^+ \sqcup M_3^-$ (Lemma E.3 item (4)).

If $\delta \in M_1^+$ I define

$$\{\delta_i\} := [\delta, \delta'] \cap M_1^+$$

with the order induced from $<$. Property (2.52) of $\{\delta_i\}$ follows from the same property of M_1^+ , which obviously holds. By construction $\delta_i \notin M_2^+, \text{Cap}[\delta_i, \beta] \geq \text{Cap}[\delta', \beta] \geq 3$. Hence condition (2.42) is satisfied.

If $\delta \in M_3^-$, then, by Lemma E.3, item (5), there is a unique $\delta'' \in M_1^+$ such that $\delta < \delta''$. I define the sequence $\{\delta_i\}$ to be $\{\delta\} \cup ([\delta'', \delta'] \cap M_1^+)$. (2.52) is automatically enforced as before.

Suppose that $\text{Cap}[\delta, \beta] \geq 3$ and $\text{Cap}[\delta', \beta] \leq 1$. I still can construct $\{\delta_i\}$. Let us choose the greatest i such that $\text{Cap}[\delta_i, \beta] = 2$. The list of intervals with $\text{Cap}[\delta, \delta'] = 2$ is given in (E.6), (E.7). All the intervals $[\delta, \delta']$ from (E.6) have $\delta' \in M_2^-$ and from (E.7) have $\delta \in M_2^+$. Then $\delta_i \in M_1^+ \cap M_2^+ = \emptyset$ or $\beta \in M_2^-$ (impossible by assumptions). Thus $\text{Cap}[\delta_i, \beta] = 2$ is ruled out in our setup.

If $\text{Cap}[\delta, \beta] \leq 1$, then any maximal totally ordered subset of $[\delta, \delta']$ can be used as $\{\delta_i\}$. ■

Corollary 2.53 *Let $[\delta, \beta] \supset [\delta', \beta]$ be intervals satisfying (2.42) such that $\delta' \in M_1^+$. Then there is a sequence of Gorenstein algebras and surjective maps*

$$A[\delta', \beta] = A[\delta_n, \beta] \xleftarrow{p_{n-1}} A[\delta_{n-1}, \beta] \xleftarrow{p_{n-2}} \cdots \xleftarrow{p_1} A[\delta_1, \beta] = A[\delta, \beta]$$

such that $\delta_i < \delta_{i+1}, \dim_{K_{rull}} A[\delta_i, \beta] = \dim_{K_{rull}} A[\delta_1, \beta] - i + 1$ and $\lambda^{\delta_i} \in \text{Ker } p_i$.

3 Digression to Cohen-Macaulay algebra

In this section I collected some useful facts about Cohen-Macaulay and Gorenstein algebras. Though the subsequent sections rely strongly on the results presented here, the reader might want to skip this

technical section at the first reading.

3.1 Definition of the Thom class th

Let

$$\mathfrak{p} : R \rightarrow S \tag{3.1}$$

be a surjective homomorphism of graded Cohen-Macaulay algebras. I assume that R and S are generated by elements in the first graded component as all the algebras that will appear in this paper. R and S are modules over the polynomial algebra P , whose first component P_1 is isomorphic to R_1 . More precisely, $R = P/\mathfrak{r}_R, S = P/\mathfrak{r}_S$. By Definition 2.36, the canonical modules ω_R and ω_S can be constructed by using the minimal free resolutions over P

$$\begin{aligned} R &\leftarrow F_0(R) \leftarrow \cdots \leftarrow F_{s-\dim(R)}(R) \leftarrow 0, \\ S &\leftarrow F_0(S) \leftarrow \cdots \leftarrow F_{s-\dim(R)+\mathrm{codim}(S)}(S) \leftarrow 0, \\ \mathrm{codim}(R, S) &:= \dim(R) - \dim(S). \end{aligned} \tag{3.2}$$

Here \dim stands for the Krull dimension and $s = \dim_{\mathbb{C}} R_1$. I denote by F^* the dual of a free finite rank P -module F . The cohomology of the dual complexes $F^{*i}(R)$ and $F^{*i}(S)$ are nonzero only in degrees $s - \dim(R), s - \dim(S)$ and coincide with the canonical modules ω_R and ω_S . By construction $F^i(\omega_R) = F^{*(s-\dim(R)-i)}(R)$, $F^i(\omega_S) = F^{*(s-\dim(S)-i)}(S)$ are resolutions of the canonical modules.

The map of algebras \mathfrak{p} defines the map of P -modules. It induces the map of the free resolutions

$$F^\bullet(R) \xrightarrow{\mathfrak{p}^\bullet} F^\bullet(S) \tag{3.3}$$

and the adjoint map between the dual complexes

$$F^\bullet(\omega_S)[\mathrm{codim}] \xrightarrow{\mathfrak{p}^{\bullet\bullet}} F^\bullet(\omega_R), \tag{3.4}$$

which I interpret as the element

$$\mathrm{th}_{\mathfrak{p}} \in \mathrm{Ext}_P^{\mathrm{codim}}(\omega_S, \omega_R). \tag{3.5}$$

In (3.5) S, R appear in the opposite order than in (3.1). This is why I call $\mathrm{th}_{\mathfrak{p}}$ a map in the "wrong direction".

3.2 The map \mathfrak{n}

The map \mathfrak{n} that will be described here, is another example of a "wrong direction" homomorphism.

Fix direct sum decomposition

$$U + U' = V$$

and a generator n of $\mathrm{Tor}_{\dim U'}^{\mathbb{C}[U']}(\mathbb{C}, \mathbb{C}) \cong \Lambda^{\dim U'}(U')^*$. Let M be a $\mathbb{C}[U]$ -module. I will be interested in a map

$$n : \mathrm{Tor}_j^{\mathbb{C}[U]}(M, \mathbb{C}) \rightarrow \mathrm{Tor}_{j+\mathrm{codim} U}^{\mathbb{C}[V]}(M, \mathbb{C}) \quad (3.6)$$

defined with a help of the Künneth decomposition

$$\mathrm{Tor}_k^{\mathbb{C}[V]}(M, \mathbb{C}) = \bigoplus_{k=i+j} \mathrm{Tor}_i^{\mathbb{C}[U]}(M, \mathbb{C}) \otimes \mathrm{Tor}_j^{\mathbb{C}[U']}(\mathbb{C}, \mathbb{C}).$$

The map is a composition of the embedding $\mathrm{Tor}_j^{\mathbb{C}[U]}(M, \mathbb{C}) \subset \mathrm{Tor}_j^{\mathbb{C}[V]}(M, \mathbb{C})$ with the multiplication on n -the generator of $\mathrm{Tor}_{\dim U'}^{\mathbb{C}[U']}(\mathbb{C}, \mathbb{C})$. Construction extends in the standard way from the category of modules $\mathrm{Mod}_{\mathbb{C}[U]}$ to the derived category $D(\mathrm{Mod}_{\mathbb{C}[U]})$.

3.3 The general properties of th

Besides the general properties of the Thom class, this section contains results of some elementary computations with th .

Proposition 3.7 *Let $R \xrightarrow{p} S \xrightarrow{q} Q$ be surjective homomorphisms of Cohen-Macaulay graded algebras. The elements $\mathrm{th}_q \in \mathrm{Ext}^{\mathrm{codim}(Q,S)}(\omega_Q, \omega_S)$, $\mathrm{th}_p \in \mathrm{Ext}^{\mathrm{codim}(S,R)}(\omega_S, \omega_R)$, $\mathrm{th}_{q \circ p} \in \mathrm{Ext}^{\mathrm{codim}(Q,R)}(\omega_Q, \omega_R)$ satisfy*

$$\mathrm{th}_{q \circ p} = \mathrm{th}_p \cup \mathrm{th}_q.$$

Proof. Follows from functorial properties of dualization. ■

The following theorem from [29] describes relation between canonical modules (Definition 2.36) for a pair Cohen-Macaulay graded rings. In the following, $\overline{x}M$ will stand for $\sum_{i=1}^m x_i M$ where $\overline{x} = (x_1, \dots, x_n) \subset R$ is a finite sequence and M is an R -module.

Theorem 3.8 ([29] 11.35) *Let R be a Cohen-Macaulay local ring and \overline{x} a regular sequence in R . If ω_R is a canonical module for R , then $R/(\overline{x})$ is Cohen-Macaulay and*

$$\omega_R/\overline{x}\omega_R \cong \omega_{R/(\overline{x})}.$$

The theorem obviously remain valid for graded rings.

The next two proposition describes $\text{th}_{\mathfrak{p}}$ in some elementary situations. In the first proposition the homomorphism \mathfrak{p} is induced by the restriction on a hypersurface. In the second \mathfrak{p} is an embedding of algebras of a special kind. Maps \mathfrak{p} that will appear in this paper will be factored into such elementary maps.

Proposition 3.9 1. *R regular homogeneous element x in a graded Cohen-Macaulay algebra R defines the homomorphism $\mathfrak{p} : R \rightarrow S = R/xR$. Then $\text{th}_{\mathfrak{p}}$ (3.5) is the class of extension*

$$0 \rightarrow \omega_R \rightarrow \omega_R \rightarrow \omega_S \rightarrow 0$$

of the dualizing modules. The map $\omega_R \rightarrow \omega_R$ is multiplication on x .

2. *If R is Gorenstein and $\bar{x} = (x_1, \dots, x_n)$ is regular, then $R/\bar{x}R$ is Gorenstein.*

Proof.

1. By abuse of notations, I denote the lift of $x \in R$ to P by the same symbol. The operator of multiplication on x in R will be denoted by \mathfrak{w} . The complex (3.2) is quasi isomorphic to the cone $C^\bullet(\mathfrak{w}^\bullet)$. Dualization transforms the canonical map $F^\bullet(R) \rightarrow C^\bullet(\mathfrak{w}^\bullet)$ to $\delta : C^\bullet(\mathfrak{w}^{*\bullet}) \rightarrow F^\bullet(\omega_R)[1]$, where $\mathfrak{w}^{*\bullet}$ is the operator of multiplication on x in $F^\bullet(\omega_R)$. The map δ is the boundary map in the distinguished triangle

$$F^\bullet(\omega_R) \rightarrow F^\bullet(\omega_R) \rightarrow C^\bullet(\mathfrak{w}^{*\bullet}) \rightarrow F^\bullet(\omega_R)[1].$$

By definition, the element

$$\delta \in \text{Hom}_P(C^\bullet(\mathfrak{w}^{*\bullet}), F^\bullet(\omega_R)[1]) = \text{Ext}_P^1(\omega_S, \omega_R)$$

is equal to $\text{th}_{\mathfrak{p}}$. In the last equality I used that R and S are Cohen-Macaulay. I also used Theorem 3.8. The triangle is quasi isomorphic to

$$\omega_R \rightarrow \omega_R \rightarrow \omega_S \xrightarrow{\delta} \omega_R[1].$$

In such description interpretation of δ as a class of extension becomes tautologous.

2. Follows from Proposition 3.1.19 [11].

■

Proposition 3.10 *Let R be a graded Gorenstein algebra. Suppose that annihilator $\text{Ann}(x) \neq 0$ for a homogenous $x \in R$. Suppose also that $S = R/\text{Ann}(x)$ is Gorenstein, $Q = R/(x)$ is Cohen-Macaulay, and $\dim(R) = \dim(S) = \dim(Q)$. The map $w : R \rightarrow R, a \rightarrow ax$ induces a homomorphism of R -modules which is a part of the exact sequence*

$$0 \rightarrow S \xrightarrow{w} R \rightarrow Q \rightarrow 0. \quad (3.11)$$

Denote by $p : R \rightarrow S = R/\text{Ann}(x)$ the homomorphism of algebras.

1. Under such assumptions, there is a short exact sequence of canonical modules

$$0 \leftarrow \omega_S \leftarrow \omega_R \leftarrow \omega_Q \leftarrow 0, \quad (3.12)$$

which is isomorphic to

$$0 \leftarrow S \xleftarrow{p} R \leftarrow \text{Ann}(x) \leftarrow 0. \quad (3.13)$$

2. Conversely, the short exact sequence of R -modules (3.13) gives rise to the short exact sequence of the modules

$$0 \rightarrow \omega_S \xrightarrow{\text{th}_p} \omega_R \rightarrow \omega_R/\omega_S \rightarrow 0$$

isomorphic to (3.11).

Proof. Under isomorphisms $R = P/\mathfrak{t}_R$, $S = P/\mathfrak{t}_S$, $Q = P/\mathfrak{t}_Q$ algebras R, S and Q become modules over P . By using the fact that $\omega_X = \text{Ext}_P^{s-\dim(X)}(X, P)$ for Cohen-Macaulay $X = R, S, Q$, I derive (3.12) from the segment of the long exact sequence of Ext groups.

The map $\omega_R \rightarrow \omega_S$ in (3.12) is compatible with the grading. It must, in the view of identifications $\omega_R \cong R$ and $\omega_S \cong S$, coincide (up to a multiplicative constant $c \in \mathbb{C}$) with p . Thus $\omega_Q \cong \text{Ann}(x)$.

Maps in (3.11) can be extended to maps of the minimal free P -resolutions

$$0 \rightarrow F^\bullet(S) \xrightarrow{w^\bullet} F^\bullet(R) \rightarrow F^\bullet(Q) \rightarrow 0.$$

As R and S are Gorenstein $F^{*\bullet}(S)$ and $F^{*\bullet}(R)$ are resolutions of S and R respectively. The map of complexes $w^{*\bullet}$ in

$$0 \leftarrow F^{*\bullet}(S) \xleftarrow{w^{*\bullet}} F^{*\bullet}(R) \leftarrow F^{*\bullet}(Q) \leftarrow 0$$

up to a constant should be equal to $p^{*\bullet}$. By symmetry $p^{*\bullet} = w^{*\bullet}$, which proves the last statement. ■

Let us fix isomorphisms

$$R \cong \omega_R \quad (3.14)$$

$S \cong \omega_S$ for Gorenstein R, S and interpret $\mathrm{th}_{\mathfrak{p}}$ (3.5) as

$$\mathrm{th}_{\mathfrak{p}} \in \mathrm{Ext}_P^{\mathrm{codim}}(S, R). \quad (3.15)$$

Proposition 3.16 *Let a pair of algebras R, S satisfy assumptions of Proposition 3.9 or Proposition 3.10. For any choice of isomorphism $\mathfrak{q} : R \cong \omega_R$, there is a unique isomorphism $\mathfrak{q}' : S \cong \omega_S$ which is a part of a commutative diagram in the derived category of R -modules*

$$\begin{array}{ccc} S & \xrightarrow{\mathfrak{q}'} & \omega_S \\ \downarrow \mathrm{th}_{\mathfrak{q}} & & \downarrow \mathrm{th}_{\mathfrak{q}'} \\ R[\dim(R) - \dim(S)] & \xrightarrow{\mathfrak{q}} & \omega_R[\dim(R) - \dim(S)]. \end{array}$$

The choice of an isomorphism $S \cong \omega_S$ determines an isomorphism $R \cong \omega_R$ with the same compatibility properties.

Proof. Suppose R, S satisfy conditions of Proposition 3.9. The isomorphism (3.14) induces an isomorphism

$$S = R \otimes_R S \cong \omega_R \otimes_R S = \omega_S \quad (3.17)$$

with the required properties because it identifies $0 \rightarrow R \rightarrow R \rightarrow S \rightarrow 0$ with $0 \rightarrow \omega_R \rightarrow \omega_R \rightarrow \omega_S \rightarrow 0$.

The isomorphism (3.14) in general is defined up to a multiplicative constant. If I alternatively fix an isomorphism $S \cong \omega_S$, I can determine the constant by applying (3.17) to (3.14).

Suppose that now R, S satisfy conditions of Proposition 3.10. I can read off the isomorphism $S \cong \omega_S$ as the isomorphism of submodules in R and ω_R induced by (3.14). Conversely, if I am given $S \cong \omega_S$, I can fix the ambiguity constant in (3.14) by comparing restriction of (3.14) on $S \subset R, \omega_S \subset \omega_R$ with $S \cong \omega_S$. ■

Let $\mathfrak{b} \subset S$ be a graded ideal, \mathfrak{c} its preimage in P . Composition with $\mathrm{th}_{\mathfrak{p}}$ (3.5), defines the map of local cohomology

$$\mathrm{th}_{S \leftarrow R} : H_{\mathfrak{b}}^i(S) \cong \varinjlim \mathrm{Ext}_P^i(P/\mathfrak{c}^n, S) \xrightarrow{\mathrm{th}_{\mathfrak{p}}} \varinjlim \mathrm{Ext}_P^{i+\mathrm{codim}}(P/\mathfrak{c}^n, R) \cong H_{\mathfrak{p}^{-1}(\mathfrak{b})}^{\mathrm{codim}+i}(R). \quad (3.18)$$

The isomorphisms \cong have been justified in the Theorem 7.11 [29].

Remark 3.19 *The map $\mathrm{th}_{\mathfrak{p}}$ commutes with the action of the algebra P . Moreover the action of P on $\mathrm{Ext}_P^i(P/\mathfrak{c}^n, S)$ factors through S . Let \mathfrak{r} be the kernel of projection map $R \rightarrow S$. Then the image of $\mathrm{th}_{S \leftarrow R}$ in $H_{\mathfrak{p}^{-1}(\mathfrak{b})}^{\mathrm{codim}+i}(R)$ is annihilated by \mathfrak{r} .*

In our applications, I will also be interested in the cohomology of the double complex (see Appendix A for notations)

$$\mathcal{T}_R^i(M) := \bigoplus_{k=j-i} B_i(\Gamma_{\mathfrak{a}} I_R^j(M), \{x^s\})$$

where $I^\bullet(M)$ is an injective resolution, $\{x^s\}$ are the homogeneous degree one generators of R . Presentation $R = P/\mathfrak{r}_R$ and isomorphism (3.18) lets to identify cohomology $H^i(\mathcal{T}_R(M))$ with the cohomology of

$$\mathcal{T}_P^k(\mathfrak{c}, M) = \bigoplus_{k=j-i} B_i(\Gamma_{\mathfrak{c}} I_P^j(M), \{x^s\}). \quad (3.20)$$

Now $\{x^s\}$ are the generators of P and \mathfrak{c} is a preimage of \mathfrak{a} in P . Let $\mathrm{R}\Gamma_{\mathfrak{c}}$ be the right derived functor of $\Gamma_{\mathfrak{c}}$ and $\otimes_{\mathbb{C}}$ be the left derived functor of $\otimes \mathbb{C}$ in the bounded from the left derived category of $D(\mathrm{Mod}_P)$. The complex (3.20) is a representative of $\mathrm{R}\Gamma_{\mathfrak{c}}(M) \otimes_{\mathbb{C}}^L$.

A surjective graded homomorphism $R = P_1/\mathfrak{r}_R \xrightarrow{p} S = P_2/\mathfrak{r}_S$ of minimal presentations can be lifted to a surjective graded map $\mathfrak{p} : P_1 \rightarrow P_2$ of polynomial algebras. Let us choose an isomorphism $P_1 \cong P_2 \otimes P_2'$. Following Section 3.2 I use construction (3.6) to define

$$\mathfrak{n} : H^k \mathcal{T}_{P_2}(S) \rightarrow H^{k-(\dim P_1 - \dim P_2)} \mathcal{T}_{P_1}(S).$$

Note that now I am using cohomological grading, which is responsible for the negative shift. The map

$$\mathrm{th}_{\mathfrak{p}} : H^k \mathcal{T}_{P_1}(S) \rightarrow H^{k+\dim R - \dim S} \mathcal{T}_{P_1}(R)$$

is induced by (3.15) if I think about it as a map in derived category. A composition of the last two maps

$$H^k \mathcal{T}_{P_2}(S) \rightarrow H^{k+(\dim R - \dim S) - (\dim P_1 - \dim P_2)} \mathcal{T}_{P_1}(R) \quad (3.21)$$

will be denoted by $\mathrm{th}'_{\mathfrak{p}}$.

3.4 Thom class and the group action

In our application, the algebras will be equipped with the action of some symmetry group. In this section, I will discuss commutation relations of th with this group action. The reader may wish to consult [47] about the general theory of resolutions of sheaves equivariant with respect to a group scheme action. In our application, when the group Aut is reductive and the ground scheme is $\mathrm{Spec} \mathbb{C}$ much of the theory becomes trivial. If we assume that the map (3.1) commutes with the Aut -action, the general theory immediately implies that the map of complexes (3.3) can be chosen to be Aut -equivariant and $\mathrm{th}_{\mathfrak{p}}$ (3.5)

is *Aut*-invariant.

In the graded Gorenstein case, the generating space
(3.22)
of the canonical module ω_R is a one-dimensional representation χ'_R of *Aut*.

Note that $\chi'_{R,P}$ depends on the presentations $R = P/\mathfrak{r}_R$. To make the definition of χ'_R less presentation-dependent, I define

$$\chi_{R,P} := \chi'_{R,P} / \det P, \quad (3.23)$$

$$\det P := \chi'_{\mathbb{C},P}. \quad (3.24)$$

Proposition 3.25 *The P -module structure on the algebra $R = P/\mathfrak{r}$ can be extended to the structure of $P \otimes P'$ -module, where variables $x_1, \dots, x_n \in P' = \mathbb{C}[x_1, \dots, x_n]$ act trivially on R . Then*

$$\chi_{R,P} = \chi_{R,P \otimes P'}. \quad (3.26)$$

Proof. The $P \otimes P'$ -resolution of R is the tensor product of the P -resolution $F_\bullet(R)$ and the Koszul complex $K(P')$. Thus $\chi'_{R,P \otimes P'} = \chi'_{R,P} \chi'_{\mathbb{C},P'}$, which implies (3.26). ■

From the point of view of representation theory, we have an isomorphism

$$\omega_R = R \otimes \chi'_R.$$

Proposition 3.27 *Let us assume that $p : R \rightarrow S$ is a surjective map of degree-one generated Gorenstein algebras and $x \in R$ has degree one.*

1. *After identification of $\text{Ext}_P^{\text{codim}}(\omega_S, \omega_R)$ with $\text{Ext}_P^{\text{codim}}(S, R)$ the group *Aut* acts on $\text{th}_{\mathfrak{p}}$ through the character*

$$\chi_{S \leftarrow R} := \chi_S \chi_R^{-1}. \quad (3.28)$$

2. *If*

$$gx = \chi_x(g)x \quad (3.29)$$

for degree-one x from Proposition 3.9, then $\chi_{S \leftarrow R} = \chi_x^{-1}$.

3. *If x from Proposition 3.10, $\deg x = 1$ satisfies (3.29), then $\chi_{S \leftarrow R} = \chi_x$.*

Proof. Item (1) is obvious. To prove item (2) I choose a minimal free P -resolution $R \leftarrow F_\bullet$. I also assume that the maps commute with Aut -action. Under my assumptions $F_\bullet/(x)$ is a minimal $P/(x)$ resolution of S . Let c be the generator of the top degree group F_d . c also generated $F_d/(x)$. From this I conclude that $\chi'_S = \chi'_R$. Also $\det(P/(x)) = \chi_x \det(P)$ and

$$\chi_S = \chi'_S / \det(P/(x)) = \chi_x^{-1} \chi'_R / \det(P) = \chi_x^{-1} \chi_R$$

Item 3 follows from Proposition 3.10. ■

3.5 An interpretation of the local cohomology as a Tor functor

In this section, I give a reformulation of local cohomology in terms of Tor-functors. Such reformulation has an advantage because projective resolutions are much more accessible than their injective analogs.

Proposition 3.30 1. Let $\mathfrak{c} \subset R$ be an ideal in a finitely generated graded commutative algebra over \mathbb{C} . Fix a polynomial algebra $P = \mathbb{C}[Y \oplus Z]$ and a graded surjective homomorphism

$$\mathfrak{w} : P \rightarrow R$$

such that $\mathfrak{w}(ZP) = \mathfrak{c}$. The homomorphism \mathfrak{w} makes R into a P -module. Let

$$\{F_i = P \otimes V_i | i = 0, \dots, d\} \tag{3.31}$$

be its minimal P -resolution. Then in the notations of Appendix C, there is an isomorphism

$$H_{\mathfrak{c}}^i(R) \cong H_{\dim Z - i}(F \otimes_P \mathbb{C}[Y] \otimes \mathbb{C}[Z]^{-1}). \tag{3.32}$$

In particular

$$H_{\mathfrak{c}}^i(R) \cong \text{Tor}_{\dim Z - i}^P(R, \mathbb{C}[Y] \otimes \mathbb{C}[Z]^{-1}).$$

2. Let Aut be an algebraic reductive group of automorphisms of R and P . Suppose that \mathfrak{w} is Aut -map and $Z \subset Y + Z$ is a pair of Aut -invariant subspaces inside of P . Then (3.32) induces an isomorphism of Aut -representations.

Proof. I will use the complex $K(R, \mathfrak{w}(\bar{x}), n)$ (\bar{x} is a basis for Z), justified by Proposition A.5, for computation of the local cohomology. For this purpose I replace R with F_\bullet . This way I get a F_i -valued Koszul complex $F_i \otimes \Lambda^j[Z]$. It is a bi-complex. The canonical map

$$\lim_{\substack{\longrightarrow \\ n}} K^{\dim Z}(F_i, \bar{x}, n) \rightarrow \mathbb{C}[Y] \otimes \mathbb{C}[Z]^{-1} \otimes_P F_i = \mathbb{C}[Y] \otimes \mathbb{C}[Z]^{-1} \otimes V_i$$

(an obvious generalization of (A.9)) defines a map of complexes

$$B^k = \bigoplus_{k=j-i} \lim_{\xrightarrow{n}} K^j(F_i, \bar{x}, n) \rightarrow \mathbb{C}[Y] \otimes \mathbb{C}[Z]^{-1} \otimes V_{\dim Z - k}. \quad (3.33)$$

The spectral sequence of B collapses in the first page, which contains only one row of groups

$$\lim_{\xrightarrow{n}} H^{\dim Z} K(F_i, \bar{x}, n) = \mathbb{C}[Y] \otimes \mathbb{C}[Z]^{-1} \otimes V_i$$

(see Example A.10). It proves that (3.33) is a quasi-isomorphism.

Lemma 3.34 *In the above assumptions R admits a finite Aut-equivariant P -resolution. Moreover any Aut-equivariant map $M \rightarrow N$ between finitely generated R modules admits an Aut-equivariant lift to a map between P -resolutions.*

Proof. It can be proven directly or deduced from the infinitely stronger results [47] ■

If we use Ext definition (A.1) of local cohomology the previous proof can be repeated. The advantage of (A.1) is that the limiting groups are representations of the symmetry group of the pair $\mathfrak{c} \subset R$. This automatically implies that all the identifications used in my proof are compatible with the Aut-action. Hence (3.32) is a Aut-map.

■

Remark 3.35 *The action of group of automorphisms Aut of the algebra R typically lifts to a formal group action on $H_{\mathfrak{c}}^i(R)$. In zero characteristics, it is encoded by the Lie algebra $\text{Lie}(G)$ representation in $H_{\mathfrak{c}}^i(R)$. If, however, a subgroup $H \subset G$ is a symmetry of the pair $\mathfrak{c} \subset R$, then restriction of representation of $\text{Lie}(G)$ on $\text{Lie}(H)$ becomes integrable (see [30] for details).*

3.6 Some remarks on the Poincaré pairing

The class of Gorenstein algebras is a source of various cohomology groups with the nondegenerate Poincaré pairing. All the results in this paper are the expansions of this observation. This section contains some general results concerning such pairings. It starts with the theorem of Avramov and Golod.

In this section, I assume that $R = P/\mathfrak{r}, \mathfrak{r}$ is graded, R is Gorenstein, $P = \mathbb{C}[U]$.

Proposition 3.36 *The cohomology of the Koszul complex $B(R)$ (see notation convention 3) are nonzero in the range from 0 to d*

$$d := \dim_{\mathbb{C}} U - \dim R, \quad (3.37)$$

$H_d(B) \cong \mathbb{C}$. The product

$$V_i \otimes V_{d-i} \rightarrow V_d, V_i = H_i(B)$$

defines a nondegenerate pairing. In fact, nondegeneracy of the pairing is equivalent to Gorenstein property of R .

$$H_i(B) \cong \text{Tor}_i^P(R, \mathbb{C}) \quad (3.38)$$

Proof. The proof is a trivial application of the main theorem in [3]. The only additional work that is required, and which I omit, is a consistent insertion of the word "graded" in the original proof because Avramov and Golod proved their theorem for Gorenstein local rings. Isomorphism $H_i(B) \cong \text{Tor}_i^P(R, \mathbb{C})$ immediately follows if I resolve \mathbb{C} with the standard Koszul resolution and apply $R \otimes_P ?$. ■

Multiplication $R \otimes R \rightarrow R$ induces a P -bilinear map on the minimal resolution (3.31)

$$\times : F_\bullet \otimes F_\bullet \rightarrow F_\bullet. \quad (3.39)$$

Remark 3.40 The induced map on $V_i = F_i \otimes_P \mathbb{C}$: $V_i \otimes V_j \rightarrow V_{i+j}$ is identified by (3.38) with the product $H_i(B) \otimes H_j(B) \rightarrow H_{i+j}(B)$.

By definition (3.31), V_i is a linear subspace F_i . It is not true in general that $\times(V_i \otimes V_j)$ contains in $V_{i+j} \subset F_{i+j}$. Suppose, however, that $i + j = d$ (3.37). Then

$$\begin{aligned} \times(V_i \otimes V_j) &\subset V_d \subset F_d \\ V_d &\xrightarrow{\mu} \mathbb{C}. \end{aligned} \quad (3.41)$$

This immediately follows from the \mathbb{C}^\times -grading considerations and the fact that F_i are a free graded cyclic modules. For this reason the pairing

$$\begin{aligned} \times : V_i \otimes V_{d-i} &\rightarrow V_d \\ \text{defined on the level of chains} &\text{coincides with the nondegenerate} \\ \text{(Proposition 4.17) cohomological pairing.} \end{aligned} \quad (3.42)$$

I will use the following proposition as a rudimental dictionary between the languages of local algebra and Tor-functors.

Proposition 3.43 $\mathfrak{c}, \mathfrak{e} \subset R$ be two graded ideals such that $\mathfrak{w}(ZP) = \mathfrak{c}$, $\mathfrak{w}(WP) = \mathfrak{e}$, $U = Y \oplus Z \oplus W$. Under identification (3.32), the product map

$$H_{\mathfrak{c}}^i(R) \otimes H_{\mathfrak{c}}^i(R) \rightarrow H_{\mathfrak{c}+\mathfrak{c}}^{i+j}(R) \quad (3.44)$$

coincides with the map

$$\begin{aligned} & \mathrm{Tor}_{\dim Z - i}^P(R, \mathbb{C}[Y \oplus W] \otimes \mathbb{C}[Z]^{-1}) \otimes \mathrm{Tor}_{\dim W - j}^P(R, \mathbb{C}[Y \oplus Z] \otimes \mathbb{C}[W]^{-1}) \rightarrow \\ & \rightarrow \mathrm{Tor}_{\dim Z + \dim W - i - j}^P(R, \mathbb{C}[Y] \otimes \mathbb{C}[Z \oplus W]^{-1}). \end{aligned}$$

It is induced by (cf. (C.6))

$$\begin{aligned} & \times : (V_i \otimes \mathbb{C}[Y \oplus W] \otimes \mathbb{C}[Z]^{-1}) \otimes (V_j \otimes \mathbb{C}[Y \oplus Z] \otimes \mathbb{C}[W]^{-1}) \rightarrow \\ & V_{i+j} \otimes \mathbb{C}[Y] \otimes \mathbb{C}[Z \oplus W]^{-1}. \end{aligned} \quad (3.45)$$

Proof. Fix a basis \bar{x} for Z and \bar{y} for W . An isomorphism

$$K(P, \bar{x}, n) \otimes K(P, \bar{y}, n) \rightarrow K(P, (\bar{x}, \bar{y}), n)$$

can be used to define the product

$$\times : K(F_{\bullet}, \bar{x}, n) \otimes K(F_{\bullet}, \bar{y}, n) \rightarrow K(F_{\bullet}, (\bar{x}, \bar{y}), n). \quad (3.46)$$

By the definition of local cohomology which uses Koszul complexes Appendix A, the map (3.46) induces (3.44). The map (3.33) gives identification of (3.46) with (3.45). ■

4 Finite approximations to the space of states

In this section, the cohomology group $H_{\mathcal{Z}[(0)^0, \delta']}^i(\mathcal{Z}[\delta, \delta'], \mathcal{O})$, the building block for the space of states, will be examined in some details.

$\mathcal{Z}[\delta, \delta']$ is an affine scheme. For such schemes most of the sheaf-theoretic constructions reduce to constructions from the commutative algebra.

Here is the precise statement concerning local cohomology of sheaves. Fix a module M over a ring R . Denote by \widetilde{M} the corresponding quasi-coherent sheaf on $\mathrm{Spec} R$.

Theorem 4.1 ([29] Theorem 12.47). *Let \mathfrak{c} be an ideal in a Noetherian ring R . Denote $\mathrm{Spec} R$ by X and the closed set $V(\mathfrak{c}) \subset X$ by Z . For each R -module M and an integer j*

$$H_{\mathfrak{c}}^j(M) \cong H_Z^j(X, \widetilde{M}).$$

From this theorem I infer that

$$H_{\mathcal{Z}[(0)^0, \delta']}^i(\mathcal{Z}[\delta, \delta'], \mathcal{O}) = H_{\mathfrak{a}}^i[\delta, \delta']$$

where

$$H_{\mathfrak{a}}^i[\delta, \delta'] := H_{\mathfrak{a}}^i(A[\delta, \delta']), \quad \mathfrak{a} = \mathfrak{i}[\delta, (1)^{-1}]$$

(see Definition 2.7 for notations).

Remark 4.2 *By Remarks 2.4 and 3.35 the group $\mathbb{C}^\times \times \text{Spin}(10)$ acts on $H_{\mathfrak{a}}^i[(0)^N, (1)^{N'}], N < N'$.*

Let A be the algebra based on $[\delta, \delta']$.

Convention 2 It is time to extend Convention 1 slightly and add to the list of abbreviations $P^{-1} = \mathbb{C}[\delta, \delta']^{-1}$ (see Appendix C equation C.2 for explanation of the notation).

The pairing (1.8) in my finite algebraic setup becomes

$$\begin{aligned} \mathfrak{m} : H_{\mathfrak{a}}^i(A) \otimes H_{\mathfrak{b}}^j(A) &\rightarrow H_{\mathfrak{a}+\mathfrak{b}}^{i+j}(A) \cong H_{\mathfrak{m}}^{i+j}(A) \\ \bigoplus_i H_{\mathfrak{m}}^i(A) &\xrightarrow{\text{res}_A} \mathbb{C}. \end{aligned} \tag{4.3}$$

The ideal

$$\mathfrak{b} = \mathfrak{i}[(0)^0, \delta']$$

defines the closed subscheme $\mathcal{Z}[\delta, (1)^{-1}] \subset \mathcal{Z}$. The sum

$$\mathfrak{m} := \mathfrak{a} + \mathfrak{b}$$

is the maximal homogeneous ideal in A . Ideal

$$\mathfrak{a}' = \mathfrak{i}[(0)^1, \delta'].$$

will be used for defining field-antifield pairing. For technical needs I will be using ideals

$$\mathfrak{f} = \mathfrak{i}[\delta, \delta']^{\leq -1}, \quad \mathfrak{f}' = \mathfrak{i}[\delta, \delta']^{\geq 0}$$

and the pairing

$$\begin{aligned} \mathfrak{m} : H_{\mathfrak{i}}^i(A) \otimes H_{\mathfrak{i}'}^j(A) &\rightarrow H_{\mathfrak{i}+\mathfrak{i}'}^{i+j}(A) \cong H_{\mathfrak{m}}^{i+j}(A) \\ \bigoplus_i H_{\mathfrak{m}}^i(A) &\xrightarrow{\text{res}_A} \mathbb{C}. \end{aligned} \tag{4.4}$$

Here are some observations about $H_{\mathfrak{a}}^i(A)$. The maps (2.10), (2.12) induce isomorphisms

$$\begin{aligned}\tau : H_{\mathfrak{a}}^i[\delta, \delta'] &\cong H_{\tau(\mathfrak{a})}^i[\tau(\delta), \tau(\delta')] \\ \sigma : H_{\mathfrak{a}}^i[\delta, \delta'] &\cong H_{\mathfrak{a}'}^i[\sigma(\delta'), \sigma(\delta)] \cong H_{\mathfrak{b}}^i[\tau^{-1}\sigma(\delta'), \tau^{-1}\sigma(\delta)].\end{aligned}\tag{4.5}$$

I used that $\sigma(\mathfrak{a}) = \mathfrak{a}'$.

The simplest question that might be asked about the pairing (4.3) are:

1. What is "size" of the groups $H_{\mathfrak{m}}^i(A)$.
2. The space \mathcal{Z} is equipped with the group action, that multiplies the functional \mathbf{res} on the character.
What is this character?
3. What properties of \mathcal{Z} and its subschemes ensure nondegeneracy of the pairing $\mathbf{res}(\mathfrak{m}(a, b))$?

There are gradings on A and the derived objects associated with Aut (1.20).

The \mathbb{C}^\times -weight of a homogeneous element will be denoted by $\deg_{\mathbb{C}^\times}$.

I will also refer to \mathbb{C}^\times -weight decomposition as to \mathbb{C}^\times -grading. (4.6)

The \mathbf{T} -weight will be denoted by $\deg_{\mathbf{T}}$ and there will be the \mathbf{T} -grading.

Let us introduce notations that we will use throughout the paper.

The space of formal linear combinations of elements $\{\lambda^\alpha | \alpha \in \mathbf{N}\}$, $\mathbf{N} \subset \hat{\mathbf{E}}$ with coefficients in \mathbb{C} will be denoted by $\text{span } \langle \mathbf{N} \rangle$. To simplify the exposition, I equip $\text{span } \langle \mathbf{N} \rangle$ with the dot product.

By definition, $\{\lambda^\alpha\}$ is an orthonormal basis for $\text{span } \langle \mathbf{N} \rangle$. (4.7)

By abuse of notations, the linear span of the set

$$\text{AltN} := \left\{ \lambda_-^i = \lambda^{\alpha^+} - \lambda^{\alpha^-} \mid \rho(\alpha^\pm) = i \in \mathbb{Z}, \alpha^\pm \in \mathbf{N} \right\} \tag{4.8}$$

will be also denoted by AltN . Introduce a notation for the quotient:

$$\text{RegN} := \text{span } \langle \mathbf{N} \rangle / \text{Alt}(\mathbf{N}) \cong \text{Alt}(\mathbf{N})^\perp = \text{span} \left\langle \lambda_+^i = \sum_{\rho(\alpha)=i, \alpha \in \mathbf{N}} \lambda^\alpha \mid i \in \rho(\mathbf{N}) \right\rangle. \tag{4.9}$$

Inclusion $\mathbf{N} \subset \mathbf{N}'$ defines a linear inclusion $\text{inc} : \text{span } \langle \mathbf{N} \rangle \subset \text{span } \langle \mathbf{N}' \rangle$. Let \mathbf{pr} be the orthogonal projection $\text{span } \langle \mathbf{N} \rangle \rightarrow \text{RegN}$. The maps inc and \mathbf{pr} define inclusions

$$\begin{aligned}\text{inc} : \text{AltN} &\rightarrow \text{AltN}', \\ \mathbf{pr} \circ \text{inc} : \text{RegN} &\rightarrow \text{RegN}' \\ \text{and } \text{span } \langle \mathbf{N} \rangle &= \text{AltN} + \text{RegN} \xrightarrow{i} \text{AltN}' + \text{RegN}' = \text{span } \langle \mathbf{N}' \rangle, \\ i &:= \text{inc} \oplus \mathbf{pr} \circ \text{inc}.\end{aligned}\tag{4.10}$$

I have to alert the reader that the map $i : \text{span } \langle N \rangle \rightarrow \text{span } \langle N' \rangle$ in general *is not* equal to $\text{inc} : \text{span } \langle N \rangle \rightarrow \text{span } \langle N' \rangle$ (c.f. Example (4.53) item (3)).

Throughout this section, I will assume that all the intervals, if not mentioned otherwise, satisfy (2.42). In particular, A will be always assumed Gorenstein if not mentioned otherwise.

4.1 The structure of the groups $H_{\mathfrak{m}}^i(A)$

In this section I use notations from Section 3.4. Let A be an algebra based on the interval $[\delta, \delta']$.

I start my analysis of the structure of the group $H_{\mathfrak{m}}^i(A)$ with the answer on the question (1) page (42).

Proposition 4.11 1.

$$\begin{aligned} H_{\mathfrak{m}}^i(A) &= \{0\} \text{ if } i \neq \text{rk}(\delta, \delta') + 1, \\ H_{\mathfrak{m}}^i(A) &\cong \bigoplus_{i \geq 0} A_i^* \text{ if } i = \text{rk}(\delta, \delta') + 1. \end{aligned} \quad (4.12)$$

There is the integer $a \in \mathbb{Z}_{>0}$ (the a -invariant of A) such that the \mathbb{C}^\times -grading of A_i^ is equal to $-a - i$.*

2. *The previous statement also holds for $A[\delta, (1)^{-1}]^{\leq -1}$.*

Proof.

By Corollary 2.43, A is Gorenstein. The proof follows from Theorem 18.7 [29].

The algebra $A[\delta, (1)^{-1}]^{\leq -1}$ is a quotient $A[\delta, (3)^{-1}]/(\lambda^{(3)^{-1}})$. By Proposition 2.35 $A[\delta, (3)^{-1}]$ is Gorenstein.

Lemma 4.13 (*Proposition 35 [36]*) *For any $\delta < \delta' \in \hat{\mathbb{E}} \text{Reg}([\delta, \delta'])$ (4.9) is a regular sequence in $A[\delta, \delta']$. The statement remains valid if $[\delta, \delta']$ is replaced by a (semi-) open interval.*

By Lemma 4.13 $\lambda^{(3)^{-1}}$ is regular in $A[\delta, (3)^{-1}]$. By Proposition 3.9 $A[\delta, (3)^{-1}]/(\lambda^{(3)^{-1}})$ is Gorenstein. The second statement of the proposition also follows from Theorem 18.7 [29]. ■

Remark 4.14 In the case of algebra \mathring{A} the integer a coincides with the index $\text{ind OGr}^+(5, 10)$

The map res_A is an isomorphism

$$\text{res}_A : A_0^* \rightarrow \mathbb{C} \quad (4.15)$$

in (4.11). Thus the \mathbb{C}^\times -grading of res coincides with the a -invariant. My next objective will be to answer question (2) and to compute the character of the Aut -action (1.20) on A_0^* .

The *Aut*-character associated with A_0^* will be denoted by $a_A(t, q, z)$ and will be called the generalized a -invariant. a_A contains more information than the a -invariant:

$$a_A(t, q, z) = t^{-a} q^{-u} z^{-r}. \quad (4.16)$$

A vector r is an element of weight lattice of $\tilde{\mathbf{T}}^5$. This action on A_0^* is closely related to the action of *Aut* on the generator of the dualizing module ω_A (3.22).

Proposition 4.17 *The cohomology $H_i(B)$ of the Koszul complex $B_\bullet(A)$ (see notation convention 3) are zero outside of the range $0 \leq i \leq d = \|\delta, \delta'\| - (\text{rk}(\delta, \delta') + 1)$. The product*

$$H_i(B) \otimes H_{d-i}(B) \rightarrow H_d(B) \cong \mathbb{C}$$

defines a nondegenerate pairing. The group $H_i(B)$ is isomorphic to $\text{Tor}_i^P(A, \mathbb{C})$.

Proof. By Corollary 2.43, A is Gorenstein. The proof follows from Proposition 3.36. ■

Proposition 4.18 *Let*

$$\chi := \chi_{A,P} \quad (4.19)$$

*be the relative character (3.23) of *Aut*-action on the generating space of the dualizing module ω_A . Then the generalized a -invariant $a := a_A$ is equal*

$$a = \chi^{-1}.$$

Proof. By Proposition 3.30, $H_{\mathfrak{m}}^i(A) \cong \text{Tor}_{\|\delta, \delta'\| - i}^P(A, P^{-1})$. Let F_\bullet (3.31), $d = \|\delta, \delta'\| - \text{rk}(\delta, \delta') + 1$, be the minimal P -resolution of A , which I will be using for computation of Tor-groups. The group $\text{Tor}_d^P(A, P^{-1})$ is a subgroup in the tensor product $P^{-1} \otimes V_d$. By Proposition 4.17, the group $V_d = \text{Tor}_d^P(A, \mathbb{C}) = H_d(B)$ is one-dimensional. Denote its generator by v . The element $\lambda^{-1} \otimes v$, where

$$\lambda^{-1} := \prod_{\alpha \in [\delta, \delta']} \frac{1}{\lambda^\alpha} \in P^{-1},$$

is a cocycle in $F_\bullet \otimes_P A$. It corresponds to the generator of A_0^* in (4.12). This is true because λ^{-1} is characterized by the property that it is the only element that is annihilated by all λ^α . The element λ^{-1} spans a one dimensional *Aut*-representation (see formula (C.7)). It follows from (C.7) that under *Aut* action λ^{-1} scales by the character $\det P$ (3.24). By Definition 2.36, representation of *Aut* in V_d is dual to the representation spanned by the generator of ω_A . Thus

$$a = \chi'^{-1} \det P = \chi^{-1}.$$

■

It is clear now that it is essential for determining scaling properties of **res** to know the action of *Aut* on the generator of the dualizing module ω_A .

4.1.1 The action of **T** on the dualizing module ω_A

The plan for this section is to derive a formula for the character χ (4.19). The idea is to compute $\chi[\delta, \delta']$ inductively. Corollary 3.27 will be used as an inductive step. To put this idea to work, I have to produce a sequence of surjective homomorphisms of Gorenstein algebras

$$A_{n+1} \leftarrow A_n \leftarrow A_{n-1} \leftarrow \cdots \leftarrow A_1$$

such that $A_{i+1} \leftarrow A_i$ satisfies assumptions of Corollary 3.27, $A_1 = A[\delta, \delta']$, $A_n = A[\delta', \delta'] \cong \mathbb{C}[\lambda^{\delta'}]$ and $A_{n+1} = \mathbb{C}$.

I will give a construction of such a sequence in the series of propositions. The reader should consult Appendix E for the definition of M_i^\pm and CL^\pm .

In this paper, we will mostly encounter algebra homomorphisms $\mathbf{p} : A[X] \rightarrow A[X']$ whose domain and codomain are labeled by intervals $X' \subset X \subset \hat{E}$. Sometimes, however, it will be useful to factor \mathbf{p} into simpler homomorphisms $A[X] \xrightarrow{\mathbf{p}'} A[Y] \xrightarrow{\mathbf{p}''} A[X']$ with $X' \subset Y \subset X$, such that Y is a semi-interval.

Here is an example of such factorization. Fix

$$\gamma \leq \gamma' < \beta \text{ and } |CL^-(\gamma)| = 1. \quad (4.20)$$

I factor the embedding

$$[\gamma, \beta] \xrightarrow{\mathbf{p}} [\gamma', \beta] \text{ into } [\gamma, \beta] \xrightarrow{\mathbf{p}'} (\gamma, \beta) \xrightarrow{\mathbf{p}''} [\gamma', \beta]. \quad (4.21)$$

I generalize this in the following construction.

Construction 4.22 Fix a system of embedded intervals

$$[\delta_1, \beta] \xrightarrow{\mathbf{p}_1} [\delta_2, \beta] \supset \cdots \xrightarrow{\mathbf{p}_{n-1}} [\delta_n, \beta].$$

I expand it by applying (4.21) to intervals based on $\gamma = \delta_i, \gamma' = \delta_{i+1}$ repeatedly whenever (4.20) is satisfied. In notations (4.21) $\mathbf{p} = \mathbf{p}_i$, $\mathbf{p}' = \mathbf{p}'_i$, $\mathbf{p}'' = \mathbf{p}''_i$. This way I get a sequence of (semi)-intervals

$$[\delta, \beta] = \Delta_1 \supset \cdots \supset \Delta_{n'} = [\delta', \beta]. \quad (4.23)$$

Lemma 4.24 1. *The homomorphism*

$$\begin{aligned} \mathbf{p} : A[\delta, \beta] &\rightarrow A[\delta', \beta] \text{ such that } \delta \leq \delta', \text{CL}^-(\delta) = 2, 3, \text{ and} \\ \delta, \delta' &\in M_1^+ \sqcup M_3^+, \beta \in M_1^- \sqcup M_3^- \end{aligned} \quad (4.25)$$

satisfies conditions of Proposition (3.9). Ker p := (λ^δ).

2. *The homomorphism*

$$\begin{aligned} \mathbf{p}' : A[\delta, \beta] &\rightarrow A(\delta, \beta] \text{ such that } |\text{CL}^-(\delta)| = 1 \text{ and} \\ \delta, &\in M_1^+ \sqcup M_3^+, \beta \in M_1^- \sqcup M_3^- \end{aligned} \quad (4.26)$$

satisfies conditions of Proposition (3.9). Ker p := (λ^δ).

The homomorphism

$$\begin{aligned} \mathbf{p}'' : A(\delta, \beta] &\rightarrow A[\delta', \beta] \text{ such that } |\text{CL}^-(\delta)| = 1, \delta \leq \delta', \text{ and} \\ \delta, \delta' &\in M_1^+ \sqcup M_3^+, \beta \in M_1^- \sqcup M_3^- \end{aligned} \quad (4.27)$$

satisfies conditions of Proposition (3.10).

In the following I will refer to (4.25, 4.26) as regular and to (4.27) as irregular homomorphisms.

Proof. As $A(\delta, \beta] \cong A[\delta, \beta]/(\lambda^\delta)$ where λ^δ is regular (Proposition 2.48) and $A[\delta, \beta]$ is Gorenstein by assumption, then, by Proposition 3.9, $A(\delta, \beta]$ is Gorenstein. I conclude that:

$$\text{If } A[\delta, \beta] \text{ is Gorenstein, then so is } A(\delta, \beta]. \quad (4.28)$$

1. By item (5) of Lemma E.3, $(\delta, \beta] = [\delta', \beta]$. By (4.28), the map $\mathbf{p} : A[\delta, \beta] \rightarrow A[\delta', \beta]$ satisfies conditions of Proposition (3.9). The same way I prove the statement about $\mathbf{p}' : A[\delta, \beta] \rightarrow A(\delta, \beta]$ when $\text{CL}^-(\delta) = 1$.
2. From items (4) and (5) of Lemma E.4, I deduce that there are two elements $\lambda^\gamma, \lambda^{\gamma'}$ in $\text{Ker } \mathbf{p}''$ belong to $\text{Ann}(\lambda^{\delta'}) \subset A(\delta, \beta]$ and γ, γ' are comparable. I can assume that $\gamma < \gamma'$. By using item (6) and the standard monomial basis in $A(\delta, \beta]$, I conclude that $\lambda^\gamma, \lambda^{\gamma'}$ generate $\text{Ker } \mathbf{p}''$. By using the standard monomial basis for $A(\delta, \beta]$ (Proposition 2.18) and item (7) from Lemma E.4, I conclude that $A(\delta, \beta]/\lambda^{\delta'} \cong A[\gamma, \beta]$. By Proposition 2.14, $A[\gamma, \beta]$ is Cohen-Macaulay. Thus \mathbf{p}'' satisfies assumptions of Proposition 3.10.

■

The following lemma is a powerful instrument. It will enables me to reduce many difficult questions related to the map $\mathbf{p} : A[\delta', \beta] \rightarrow A[\delta, \beta]$ to a series of easy questions about the factors \mathbf{p}_i in the factorization of \mathbf{p} .

Lemma 4.29 *Fix pair of intervals*

$$[\delta, \beta] \supset [\delta', \beta] \quad (4.30)$$

$\delta' \in M_1^+$ (2.40). By using Proposition 2.50, it can be completed to the sequence of embedded intervals (2.51). With the help of Construction 4.22, I expand it to the sequence of sets (4.23). Inclusions of sets (4.23) define a series of homomorphisms of algebras

$$A[\Delta_{n'}] \xleftarrow{\mathbf{p}_{n'-1}} \dots \xleftarrow{\mathbf{p}_1} A[\Delta_1]. \quad (4.31)$$

I claim that any such homomorphism $A[\Delta_i] \rightarrow A[\Delta_{i+1}]$ satisfies assumptions of Proposition 3.9 or Proposition 3.10.

Proof. By reading carefully conditions of Lemma 4.24, I conclude that $|\text{CL}^-(\delta_i)|$ has to be 1, 2 or 3, which are automatically satisfied. Thus, the statement of lemma follows from Lemma 4.24.

■

Definition 4.32 Fix $[\delta, \beta] \subset [\delta', \beta']$ such that $A[\delta, \beta], A[\delta', \beta']$ are Gorenstein (Theorem 2.35). Let $\mathbf{p} : A[\delta', \beta'] \rightarrow A[\delta, \beta]$ be projection the projection (Definition 2.7). Equation (3.4) defines the element

$$\text{th}_{\mathbf{p}} \in \text{Ext}_P^{\text{codim}}(A[\delta, \beta], A[\delta', \beta']), \text{codim} = \rho(\beta') - \rho(\beta) - \rho(\delta') + \rho(\delta), P = \mathbb{C}[\delta', \beta'] \quad (4.33)$$

By (3.18) $\text{th}_{\mathbf{p}}$ defines a map of local cohomology

$$H_{\mathfrak{c}}^i[\delta, \beta] \rightarrow H_{\mathfrak{c}}^{\text{codim}+i}[\delta', \beta']$$

$\mathfrak{c} \subset A[\delta', \beta']$ is an ideal. By abuse of notations I denote $\mathbf{p}(\mathfrak{c})$ by \mathfrak{c} .

Introduce a notation:

$$\chi'[\alpha, \beta] := \chi'_{A[\alpha, \beta], \mathbb{C}[\alpha, \beta]}, \quad \chi[\alpha, \beta] := \chi_{A[\alpha, \beta]}$$

The results of the following proposition will be used heavily in the computation of the *Aut*-weight of the pairing (1.8).

Proposition 4.34 *Fix pair of intervals $[\delta, \beta] \supset [\delta', \beta]$. The group *Aut* commutes with the homomorphism $\mathbf{p} : A[\delta, \beta] \rightarrow A[\delta', \beta]$.*

1. The element

$$\mathbf{th}_{\mathbf{p}} \in \text{Ext}_P^{\text{codim}}(A[\delta', \beta], A[\delta, \beta]), \text{codim} = \rho(\delta') - \rho(\delta), P = \mathbb{C}[\delta, \beta]$$

scales under the action of Aut . More precisely if $g = (t, q, z)$, then $\mathbf{th}_{\mathbf{p}}$ satisfies

$$\mathbf{th}_{\mathbf{p}}^g = \mathbf{th}_{\mathbf{p}} \chi[\delta', \beta](g) \chi[\delta, \beta]^{-1}(g), g \in \text{Aut}. \quad (4.35)$$

2. If $\mathbf{th}_{\mathbf{p}}$ corresponds to the elementary homomorphisms of algebras (4.25, 4.26, 4.27), then in notations (2.28)

$$\mathbf{th}_{\mathbf{p}}^g = \begin{cases} \hat{v}_{\alpha}^{-1} \mathbf{th}_{\mathbf{p}} & \mathbf{p} \text{ is the map } A[\alpha, \beta] \rightarrow A(\alpha, \beta) \text{ (4.25, 4.26)} \\ \hat{v}_{\gamma} \mathbf{th}_{\mathbf{p}} & \mathbf{p} \text{ is the map } A(\alpha, \beta] \rightarrow A[\gamma, \beta] \text{ (4.27)}. \end{cases} \quad (4.36)$$

See formula (2.28) for the character \hat{v}_{δ} . In particular, the products $\chi'(\alpha, \beta) \chi'[\alpha, \beta]^{-1}$, $\chi'[\gamma, \beta] \chi'(\alpha, \beta)^{-1}$ in (4.36) do not depend on β .

3. The character $\det \mathbb{C}[\delta, \beta]$ (3.22) satisfies

$$(\det \mathbb{C}[\delta, \beta])^{-1} = \prod_{\alpha \in [\delta, \beta]} \hat{v}_{\alpha}. \quad (4.37)$$

Proof. Follows from Corollary 3.27. The last item follow from a simple computation with the Koszul resolution for \mathbb{C} . ■

I define quantities (a_L, u_L, r_L) by the formula $t^{-a_L} q^{-u_L} z^{-r_L} := \det \mathbb{C}[\delta, \beta]$. In more details

$$a_S[\delta, \beta] = |[\delta, \beta]|, \quad u_S[\delta, \beta] = \sum_{\alpha \in [\delta, \beta]} u(\alpha), \quad z^{r_S[\delta, \beta]} = \prod_{\alpha \in [\delta, \beta]} v_{\alpha}(z)$$

The next proposition reduces computation of $\chi[\delta, \beta]$ to the problem of combinatorics.

Proposition 4.38 Fix an interval $[\delta, \beta]$.

$$L[\delta, \beta] = M \cap \text{Core}[\delta, \beta], \quad \overline{\text{Core}}[\delta, \beta] := [\delta, \beta] \setminus \text{Core}[\delta, \beta]. \quad (4.39)$$

See Lemma E.3 for definition of M and (2.20) for Core . Then

$$\chi[\delta, \beta] = \prod_{\alpha \in L[\delta, \beta]} \hat{v}_{\alpha} \prod_{\alpha \in \overline{\text{Core}}[\delta, \beta]} \hat{v}_{\alpha} \quad (4.40)$$

Proof. I leave as an exercise the trivial verifications based on Corollary 3.27 and formulas (4.35), (4.36) that:

1. If δ, δ' satisfy conditions (4.25), then

$$\chi[\delta, \beta] = \hat{v}_\delta \chi[\delta', \beta]. \quad (4.41)$$

2. If δ, δ' satisfy conditions (4.26), then

$$\chi[\delta, \beta] = \hat{v}_\delta \chi[\delta, \beta]. \quad (4.42)$$

3. If δ, δ' satisfy conditions (4.27), then

$$\begin{aligned} (\delta, \beta] &= [\delta', \beta] \cup \{\delta''\}, \text{ where } \delta'' \neq \delta', \rho(\delta'') = \rho(\delta'), \\ \chi(\delta, \beta] &= \hat{v}_{\delta'}^{-1} \chi[\delta', \beta] \text{ and } \chi[\delta, \beta] = \hat{v}_\delta \hat{v}_{\delta'}^{-1} \chi[\delta', \beta]. \end{aligned} \quad (4.43)$$

The rest of the proof is the induction on the cardinality of $\text{Core}[\delta, \beta] \cap M_1^+$, which I omit. During induction I get a presentation of $\chi[\delta, \beta]$ as a product of \hat{v}_α^\pm . The key moment is that $\hat{v}_\alpha, \alpha \in M_1^+ \setminus M$ appears in this product twice, one time in the numerator and the other in the denominator.

■

It often happens that an equation satisfied by a function is more illuminating than an explicit formula for the function. In the next proposition, I derive such an equation for $\chi[\delta, \delta']$.

Proposition 4.44 *Let us assume hypothesis of Proposition 4.38.*

1. Let $L = L[\delta, \beta]$ be as in (4.39). Then the exponents of the character $\chi[\delta, \beta] = t^{-a} q^{-u} z^{-r}$ are

$$\begin{aligned} a[\delta, \beta] &= -|L \cup \overline{\text{Core}}|, \\ u[\delta, \beta] &= - \sum_{\alpha \in L \cup \overline{\text{Core}}} u(\alpha), \\ z^{r[\delta, \beta]} &= \prod_{\alpha \in L \cup \overline{\text{Core}}} v_\alpha^{-1}(z), \end{aligned} \quad (4.45)$$

which coincide with the components of the Aut-weight of res .

2. Suppose that $\delta < (0)^l, (1)^{l-1} < \beta$. Then

$$\chi[\delta, \beta] = \chi[\delta, (1)^{l-1}] \chi[(0)^l, \beta] q^{2-4l} t^{-4}.$$

In particular, if $l = 0$, then

$$\chi[\delta, \delta'] = t^{-4} q^2 \chi[\delta, (1)^{-1}] \chi[(0)^0, \delta']. \quad (4.46)$$

3.

$$\chi[\delta, \beta] = \chi[\delta, \beta]^{\leq l} \chi[\delta, \beta]^{\geq l+1}.$$

4.

$$\begin{aligned} a[\delta, \beta] &= a[\sigma(\beta), \sigma(\delta)], & a[\tau(\delta), \tau(\beta)] &= a[\delta, \beta], \\ u[\delta, \beta] &= -u[\sigma(\beta), \sigma(\delta)], & u[\tau(\delta), \tau(\beta)] &= u[\delta, \beta] + a[\delta, \beta], \\ r[\delta, \beta] &= S(r[\sigma(\beta)], \sigma(\delta)), & r[\tau(\delta), \tau(\beta)] &= r[\delta, \beta]. \end{aligned} \tag{4.47}$$

Proof. Only the last item in the proposition requires a comment. By Lemma E.3, item 4, the set M is invariant under σ and τ . Thus σ induces a bijection between $L[\delta, \beta]$ and $L[\sigma(\beta), \sigma(\delta)]$ and τ between $L[\delta, \beta]$ and $L[\tau(\delta), \tau(\beta)]$. ■

Here is a result of computation of χ for the most elementary sets.

Proposition 4.48

$$\chi[(0)^r, (1)^r] = t^8 q^{8r}.$$

Proof. $\mathbb{C}^\times \times \mathbf{T} \times \text{Spin}(10)$ is the symmetry group of $A[(0)^r, (1)^r]$. It means that the $\tilde{\mathbf{T}}^5$ -character $\chi[(0)^r, (1)^r]$ is a restriction of one-dimensional representation of $\text{Spin}(10)$. As $\text{Spin}(10)$ is semisimple, any one-dimensional representation is trivial. The character of $\mathbb{C}^\times \times \mathbf{T}$ is computed directly with (4.40).

■

The following corollary is a combination of the results of Propositions 4.44 and 4.48.

Corollary 4.49 *Suppose $N < N'$. Then*

$$\chi[(0)^N, (1)^{N'}] = t^{8+4(N'-N)} q^{2N'^2+4N'-2N^2+4N}$$

Remark 4.50 *If I choose $[\delta, \delta']$ such that $\sigma[\delta, \delta'] = [\delta, \delta']$ for σ as in (2.11), then $\chi[\delta, \delta'](t, q, z) = \chi[\delta, \delta'](t, q^{-1}, S(z))$ and $\chi[\delta, \delta'](t, q, z)$ doesn't depend on q and z .*

4.2 The groups $H_a^i[\delta, \delta']$

In this section, I will answer question 3 from the page 42 about nondegeneracy of the pairing $\text{res}(\mathfrak{m}(a, b))$. To do this I need to explore the groups $H_a^i[\delta, \delta']$ more closely. My exposition will become more accessible with the use of certain D -modules, which I introduce in the next section.

In Sections 4.2.2, 4.2.3, 4.2.4, 4.2.5, all the intervals $[\delta, \delta']$ satisfy $\delta \leq (0)^0, (1)^{-1} \leq \delta'$ and the purity condition (2.42).

4.2.1 The auxiliary D-modules

Proposition 3.30 enables me to identify local cohomology of A with the Tor functor over the free commutative algebra P . In the isomorphism $H_{\mathfrak{c}}^i(A) = \text{Tor}_j^P(A, M)$ (3.32), the non-finitely generated P -module M depends on the presentation $A = P/\mathfrak{r}$ and the ideal \mathfrak{c} . In order to describe M in finite terms, I will introduce the algebra of differential operators $D \supset P$. The module M will become a cyclic D -module. My present goal is to describe a number of D modules that will be used in the outlined above computation of local cohomology. I will use non uniqueness of the choice of M to my advantage: one choice (type T) of M will illuminate the gradings on $H_{\mathfrak{c}}^i(A)$, the other (type S) will make $H_{\mathfrak{c}}^i(A)$ more accessible for computations. The reader may check from time to time with Appendix C for notations and some elementary constructions that will be used in the rest of the paper. I fix the interval $[\delta, \delta']$ which satisfy $\delta < (0)^0, (1)^{-1} < \delta'$, for the rest of the section so that all the constructions will be based on this choice.

The differential operators based on $\text{span} \langle [\delta, \delta'] \rangle$ (I use notations (4.7, 4.8, 4.9)) will be denoted by $D[\delta, \delta']$. With the help of the dot product (4.7) I identify λ^α with coordinates on $\text{span} \langle [\delta, \delta'] \rangle$.

It is customary in the present context to denote the dual basis to $\{\lambda^\alpha\}$ for $\text{span} \langle [\delta, \delta'] \rangle^*$ by $\{w_{\sigma(\alpha)} | \alpha \in [\delta, \delta']\}$ (the involution σ was introduced in (2.11). In the presence of the dot product, $w_{\sigma(\alpha)}$ has an interpretation of the partial derivative $\frac{\partial}{\partial \lambda^\alpha}$. The algebra $D[\delta, \delta']$ is generated by $\{\lambda^\alpha, w_{\sigma(\alpha)} | \alpha \in [\delta, \delta']\}$ that are subject to relations

$$[\lambda^\alpha, \lambda^{\alpha'}] = [w_\beta, w_{\beta'}] = 0$$

$$[\lambda^\alpha, w_\beta] = \delta_{\alpha, \sigma(\beta)}.$$

Introduce the following subsets of $[\delta, \delta']$.

$$N(\mathfrak{c}) = \begin{cases} \emptyset & \text{if } \mathfrak{c} = \{0\} \\ [\delta, (1)^{-1}] & \text{if } \mathfrak{c} = \mathfrak{a} \\ [(0)^1, \delta'] & \text{if } \mathfrak{c} = \mathfrak{a}' \\ [(0)^0, \delta'] & \text{if } \mathfrak{c} = \mathfrak{b} \\ [\delta, \delta']^{\leq -1} & \text{if } \mathfrak{c} = \mathfrak{f} \\ [\delta, \delta']^{\geq -1} & \text{if } \mathfrak{c} = \mathfrak{f}' \\ [\delta, \delta'] & \text{if } \mathfrak{c} = \mathfrak{m} \\ E & \text{if } \mathfrak{c} = \mathfrak{p}. \end{cases} \quad (4.51)$$

Sets $N(\mathfrak{c})$ satisfy

$$N(\mathfrak{m}) = [\delta, \delta'] = N(\mathfrak{a}) \cup N(\mathfrak{p}) \cup N(\mathfrak{a}'), \quad N(\mathfrak{p}) \cup N(\mathfrak{a}') = N(\mathfrak{b}).$$

Components of the orthogonal sum decomposition

$$\text{span} \langle [\delta, \delta'] \rangle = \text{Alt}[N(\mathfrak{a})] + \text{Reg}[N(\mathfrak{a})] + \text{Reg}[N(\mathfrak{b})] + \text{Alt}[N(\mathfrak{b})]$$

were defined in (4.8) and (4.9). Maps (4.10) induce inclusion of algebras

$$\begin{aligned} \mathbf{k}^{\text{Alt}} : D(\text{Alt}[\delta, \gamma]) &\rightarrow D(\text{Alt}[\delta', \gamma]), \\ \mathbf{k}^{\text{Reg}} : D(\text{Reg}[\delta, \gamma]) &\rightarrow D(\text{Reg}[\delta', \gamma]), \\ \delta' &< \delta < \gamma. \end{aligned} \tag{4.52}$$

To become more familiar with functorial properties of Alt and Reg (4.10), let us consider an example:

Example 4.53 1. $\text{Alt}[(0)^0, (13)^0] = 0$, $\text{Reg}[(0)^0, (13)^0] = \text{span} \langle [(0)^0, (13)^0] \rangle$.

2. The map (4.10) of $\text{Reg}[(0)^0, (13)^0]$ to $\text{Reg}[(45)^{-1}, (13)^0]$ is

$$\mathbf{i}(\lambda^{(0)^0}) = \frac{1}{\sqrt{2}}(\lambda^{(0)^0} + \lambda^{(3)^{-1}}), \quad \mathbf{i}(\lambda^{(12)^0}) = \frac{1}{\sqrt{2}}(\lambda^{(12)^0} + \lambda^{(2)^{-1}}), \quad \mathbf{i}(\lambda^{(13)^0}) = \lambda^{(13)^0}.$$

3. The map $\mathbf{i} : \text{span} \langle [(4)^{-1}, (13)^0] \rangle \rightarrow \text{span} \langle [(35)^{-1}, (13)^0] \rangle$ (4.10) transforms $\lambda^{(4)^{-1}}$ to $\frac{1}{\sqrt{2}}(\lambda^{(4)^{-1}} + \lambda^{(45)^{-1}})$.

Modules $T_{\mathfrak{c}}$ By using notations (4.51) and (C.5), I define a family $T_{\mathfrak{c}}$, $\mathfrak{c} = (0), \mathfrak{a}, \mathfrak{a}', \mathfrak{b}, \mathfrak{p}, \mathfrak{m}$ of P -modules

$$\begin{aligned} T_{(0)} &:= \mathbb{C}[\delta, \delta'] \\ T_{\mathfrak{a}} &:= \mathbb{C}[N(\mathfrak{a})]^{-1} \otimes \mathbb{C}[N(\mathfrak{p})] \otimes \mathbb{C}[N(\mathfrak{a}')] \\ T_{\mathfrak{a}'} &:= \mathbb{C}[N(\mathfrak{a})] \otimes \mathbb{C}[N(\mathfrak{p})] \otimes \mathbb{C}[N(\mathfrak{a}')]^{-1} \\ T_{\mathfrak{b}} &:= \mathbb{C}[N(\mathfrak{a})] \otimes \mathbb{C}[N(\mathfrak{b})]^{-1} \\ T_{\mathfrak{f}} &:= \mathbb{C}[N(\mathfrak{f})]^{-1} \otimes \mathbb{C}[N(\mathfrak{f}')] \\ T_{\mathfrak{f}'} &:= \mathbb{C}[N(\mathfrak{f})] \otimes \mathbb{C}[N(\mathfrak{f}')]^{-1} \\ T_{\mathfrak{m}} &:= \mathbb{C}[N(\mathfrak{m})]^{-1}. \end{aligned} \tag{4.54}$$

In the following, I occasionally will use an abbreviation

$$\mathcal{L} := \mathbb{C}[z, z^{-1}]$$

for the ring of Laurent polynomials. If N is a \mathbb{C} vector space, $N[z, z^{-1}]$ will stand for a free $\mathbb{C}[z, z^{-1}]$ -module $N \otimes \mathbb{C}[z, z^{-1}]$. A span of a set of elements $\{x_i\}$ in a $\mathbb{C}[z, z^{-1}]$ -module M will be denoted by

$\text{span}_{\mathcal{L}} \langle x_i \rangle$. A free $\mathbb{C}[z, z^{-1}]$ -module $\text{span}_{\mathcal{L}} \langle [\delta, \delta'] \rangle$ contains a submodule spanned over $\mathbb{C}[z, z^{-1}]$ by

$$\lambda^{\beta}(z) := \sum_{\beta^k \in [\delta, \delta']} \lambda^{\beta^k} z^k, \quad \beta \in E. \quad (4.55)$$

Lemma 4.56 *There is a direct sum decomposition of free $\mathbb{C}[z, z^{-1}]$ modules.*

$$\text{span}_{\mathcal{L}} \langle [\delta, \delta'] \rangle \cong \text{span}_{\mathcal{L}} \langle \lambda^{\beta}(z) \rangle \oplus \text{span}_{\mathcal{L}} \langle N(\mathbf{a}) \rangle \oplus \text{span}_{\mathcal{L}} \langle N(\mathbf{a}') \rangle. \quad (4.57)$$

Proof. The proof becomes obvious if I use the fact that the map

$$\text{span}_{\mathcal{L}} \langle [\delta, \delta'] \rangle \rightarrow \text{span}_{\mathcal{L}} \langle [\lambda^{\beta}] \rangle \quad \lambda^{\beta^k} \rightarrow \delta_{0,k} \lambda^{\beta}$$

after restriction on $\text{span}_{\mathcal{L}} \langle \lambda^{\beta}(z) \rangle$ induces an isomorphism

$$\text{span}_{\mathcal{L}} \langle \lambda^{\beta}(z) \rangle \cong \text{span}_{\mathcal{L}} \langle \lambda^{\beta} \rangle.$$

■

Corollary 4.58 *The $P[z, z^{-1}]$ module*

$$T_{\mathbf{p}} := \mathbb{C}[\lambda^{\beta}(z)]^{-1} \otimes \mathbb{C}[N(\mathbf{a})] \otimes \mathbb{C}[N(\mathbf{a}')]]$$

is free over $\mathbb{C}[z, z^{-1}]$.

A P -modules $T_{\mathbf{c}}$ is a D -modules. The elements $w_{\sigma(\alpha)}$ act by $\frac{\partial}{\partial \lambda^{\alpha}}$. The products

$$\begin{aligned} \varpi_{\mathbf{c}} &:= \prod_{\alpha \in N(\mathbf{c})} (\lambda^{\alpha})^{-1} \mathbf{c} \neq \mathbf{p}, \\ \varpi_{\mathbf{p}} &:= \prod_{\beta \in E} (\lambda^{\beta}_{[\delta, \delta']}(z))^{-1} \end{aligned}$$

are D generators for $T_{\mathbf{c}}$ and $T_{\mathbf{p}}$ respectively. Infinitesimal symmetries of the algebra A act on $T_{\mathbf{c}}$. Some of them can be integrated to an algebraic action of the corresponding Lie group. In particular, this applies to $\mathbb{C}^{\times} \times \mathbf{T}$. Introduce a function

$$s(\mathbf{c}) := |N(\mathbf{c})|. \quad (4.59)$$

Degree $\deg_{\mathbb{C}^{\times}}$ (4.6) and \mathbf{T} -weight $\deg_{\mathbf{T}}$ of the generators $\varpi_{\mathbf{c}}$ $\mathbf{c} \neq \mathbf{p}$ is equal to

$$\begin{aligned} \deg_{\mathbb{C}^{\times}} \varpi_{\mathbf{c}} &= -s(\mathbf{c}), \\ \deg_{\mathbf{T}} \varpi_{\mathbf{c}} &= - \sum_{\beta^k \in N(\mathbf{c})} k. \end{aligned}$$

The isomorphism described in the next proposition will be used in the definition of the field - antifield pairing.

Proposition 4.60 *There is an isomorphism*

$$T_{\mathfrak{m}} \xrightarrow{\phi} T_{\mathfrak{a}} \otimes_P T_{\mathfrak{p}} \otimes_P T_{\mathfrak{a}'}. \quad (4.61)$$

In the above equation I extended scalars from \mathbb{C} to $\mathbb{C}[z, z^{-1}]$ where it was necessary.

Proof. It follows from decomposition (4.57) that the right-hand-side of (4.61) is nonzero. Both sides of (4.61) are modules over $D[z, z^{-1}]$. The generators $w_{\sigma(\alpha)}$ act diagonally. The product $\varpi_{\mathfrak{a}}\varpi_{\mathfrak{p}}\varpi_{\mathfrak{a}'}$ is annihilated by $\lambda^{\beta}(z)$ and $\lambda^{\alpha}, \alpha \in N(\mathfrak{a}) \cup N(\mathfrak{a}')$. By Lemma 4.56, it is also annihilated by $\lambda^{\alpha}, \alpha \in [(0)^0, (1)^0]$. By construction, $T_{\mathfrak{m}}$ is a D -cyclic module. The easiest way to see that $\varpi_{\mathfrak{a}}\varpi_{\mathfrak{p}}\varpi_{\mathfrak{a}'}$ is a generator of the tensor product is to replace $\{w_{\sigma\beta^k}\} \subset D$ with a different set of generators $\{w_{\sigma\beta^0}, w_{\sigma\beta^k} - z^k w_{\sigma\beta^0}, k \neq 0\}$. For this new set verification is trivial. The map ϕ of cyclic $D[z, z^{-1}]$ -modules which sends $\varpi_{\mathfrak{m}}$ to $\varpi_{\mathfrak{p}}\varpi_{\mathfrak{a}}\varpi_{\mathfrak{a}'}$ is self-consistent because $\lambda^{\alpha}\varpi_{\mathfrak{m}} = 0$ are the defining relation for $T_{\mathfrak{m}}$. ■

Remark 4.62 1. *The \mathbf{T} -weight subspaces of $T_{\mathfrak{a}}$ and $T_{\mathfrak{b}}$ are finite-dimensional. \mathbf{T} -weights of $T_{\mathfrak{a}}$ are bounded from below by the weight of $\varpi_{\mathfrak{a}}$. In $T_{\mathfrak{b}}$ case the weights are bounded from above by the weight of $\varpi_{\mathfrak{b}}$.*

2. *If $\delta < (1)^{-1}, (0)^0 < \delta'$ the \mathbf{T} -weight subspaces of $T_{\mathfrak{m}}$ are infinite-dimensional. If $\delta' = \delta'^k, k < 0$ then $\varpi_{\mathfrak{m}}$ is the lowest \mathbf{T} -weight vector in $T_{\mathfrak{m}}$.*

Definition 4.63 Fix a linear space $M = \bigoplus_{u \in \mathbb{Z}} M_u, \dim M_u < \infty$. M is a *positive energy space* if $\exists u_0$ such that $M_{u_0} \neq \{0\}, M_u = \{0\}, u < u_0$. I denote u_0 by $\underline{u}(M)$. The *negative energy space* is defined similarly. The weights upper bound is denoted by $\overline{u}(M)$.

For example, $T_{\mathfrak{a}}$ is a positive energy space with respect to $\deg_{\mathbf{T}}$ -grading. $T_{\mathfrak{b}}$ is a negative energy space. The isomorphism

$$T_{\mathfrak{a}} \otimes_P T_{\mathfrak{b}} \cong T_{\mathfrak{m}}$$

follows easily from the isomorphisms $x^{-1}\mathbb{C}[x] \otimes_{\mathbb{C}[x]} \mathbb{C}[x] \cong x^{-1}\mathbb{C}[x]$, (C.5) and (4.54). I use it to define map

$$T_{\mathfrak{a}} \otimes T_{\mathfrak{b}} \rightarrow T_{\mathfrak{m}}, \quad T_{\mathfrak{j}} \otimes T_{\mathfrak{j}'} \rightarrow T_{\mathfrak{m}}, \text{ and} \quad (4.64)$$

$$T_{(0)} \otimes T_{\mathfrak{m}} \rightarrow T_{\mathfrak{m}}.$$

The residue functional

$$\text{res}_{T_{\mathfrak{m}}} : T_{\mathfrak{m}} \rightarrow \mathbb{C} \quad (4.65)$$

is

$$\text{res}_{T_m}(a) := \oint \cdots \oint a \prod_{\alpha \in [\delta, \delta']} d\lambda^\alpha.$$

res_T and the product (4.64) define the pairing:

$$T_a \otimes T_b \rightarrow \mathbb{C}, \quad T_{(0)} \otimes T_m \rightarrow \mathbb{C}, \quad (a, b)_{T_m} = \text{res}_{T_m}(ab). \quad (4.66)$$

Proposition 4.67 *The pairings (4.66) are not degenerate.*

Proof. Elements

$$\begin{aligned} \prod_{\alpha \in N(\mathfrak{b})} (\lambda^\alpha)^{k_\alpha} \prod_{\beta \in N(\mathfrak{a})} (\lambda^\beta)^{-1-n_\beta} &\in T_a, \\ \prod_{\alpha \in N(\mathfrak{b})} (\lambda^\alpha)^{-1-k_\alpha} \prod_{\beta \in N(\mathfrak{a})} (\lambda^\beta)^{n_\beta} &\in T_b \end{aligned}$$

form a pair of dual bases in modules T_a and T_b respectively. Similar bases exist in $T_{(0)}$ and T_m . ■

Modules S_a, S_b Description of modules of type S is less straightforward than of type T . Besides giving the definition, I will devote some efforts exploring properties $S[\delta, \delta']$ as a function of the interval.

Definition 4.68 $K^{\leq n} := \{\alpha \in K | \rho(\alpha) \leq n\}$. This construction will also be used with other binary relations $=, <, >, \geq$. Here ρ is the function (B.3). I will also use partial order in \hat{E} to define $K^{\leq \beta} := \{\alpha \in K | \alpha \leq \beta\}$ and similar constructs based on $<, >, \geq$.

By using Alt (4.8) and Reg (4.9) constructions and a finite subset $K \subset \hat{E}$, I define

$$\begin{aligned} D_-[K] &:= D(\text{Alt } K), \\ D_+[K] &:= D(\text{Reg } K), \\ \mathbb{C}_-[K] &:= \mathbb{C}[\text{Alt } K], \\ \mathbb{C}_+[K] &:= \mathbb{C}[\text{Reg } K], \\ \mathbb{C}_+[K]^{-1} &:= \mathbb{C}[\text{Reg } K]^{-1}. \end{aligned} \quad (4.69)$$

The modules $S_a[K], S_{a'}[K]$ over

$$D[K] \cong D_-[K^{\leq(1)^{-1}}] \otimes D_+[K^{\leq(1)^{-1}}] \otimes D[K^{\geq(0)^0}]$$

are the tensor products

$$\begin{aligned} S_a[K] &:= \mathbb{C}_-[K^{\leq(1)^{-1}}] \otimes \mathbb{C}_+[K^{\leq(1)^{-1}}]^{-1} \otimes \mathbb{C}[K^{\geq(0)^0}], \\ S_{a'}[K] &:= \mathbb{C}[K^{\leq(1)^0}] \otimes \mathbb{C}_-[K^{\geq(0)^1}] \otimes \mathbb{C}_+[K^{\geq(0)^1}]^{-1}. \end{aligned} \quad (4.70)$$

Fix finite $Q \subset \hat{E}$ such that $Q^{\geq(0)^0} = Q'^{\geq(0)^0}$. Let us apply construction (C.4) to spaces $V = \text{Alt } Q^{\leq(1)^{-1}}, V' = \text{Alt } Q'^{\leq(1)^{-1}}, U = \text{Reg } Q^{\leq(1)^{-1}}, U' = \text{Reg } Q'^{\leq(1)^{-1}}$. The maps $\mathbf{a} = \text{inc}, \mathbf{b} = \text{pr} \circ \text{inc}$ are taken from 4.10. After taking isomorphisms (4.70) into account, I assemble a map

$$\begin{aligned} \mathbf{i} : S_{\mathbf{a}}[Q] &\cong S_{\mathbf{a}}[Q^{\leq(1)^{-1}}] \otimes S_{\mathbf{a}}[Q^{\geq(0)^0}] \longrightarrow S_{\mathbf{a}}[Q'^{\leq(1)^{-1}}] \otimes S_{\mathbf{a}}[Q'^{\geq(0)^0}] \cong S_{\mathbf{a}}[Q'], \\ \mathbf{i} &:= \mathbf{j} \otimes \text{id}. \end{aligned} \tag{4.71}$$

The component \mathbf{j} is $\text{inc} \otimes (\text{pr} \circ \text{inc})$.

In the notations of Appendix C the $D[K]$ -generator $\varpi_{\mathbf{a}}(K)$ of $S_{\mathbf{a}}[K]$ is

$$\varpi_{\mathbf{a}}(K) = \prod_{\lambda_+^i \in \text{Reg } K^{\leq(1)^{-1}}} \frac{1}{\lambda_+^i}.$$

From now on to the end of the section I assume that $\delta, \delta' \in M_1^+$ (2.40) and $\delta, \delta' < (0)^0, (1)^{-1} < \gamma$. My next plan is to modify inclusions (4.71).

Definition 4.72 Suppose $\delta' \triangleleft \delta$ and $[\delta', \gamma] \supset [\delta, \gamma]$

1. If $\delta' \in M_2^- \subset M_1^+$ (Definition E.2)

$$\mathbf{k}(a) := \lambda^{\delta'} \mathbf{i}(a), \tag{4.73}$$

(c.f. the group of homomorphisms (4.27) and the irregular map Th (6.3)).

2. If $\delta' \in M_1^+ \setminus M_2^-$

$$\mathbf{k}(a) := \mathbf{i}(a) \tag{4.74}$$

(c.f. the group of homomorphisms (4.25,4.26) and the regular map Th (6.2)).

Define the element

$$m_{\delta} := \prod_{\alpha \in M_2^- \cap N(\mathbf{a})} \lambda^{\alpha}.$$

Definition 4.75 \mathbf{k} -maps (4.52) and inclusions $[\delta, \gamma] \subset [\delta', \gamma]$ $\delta, \delta' \in M_1^+$ define inclusions of algebras $D(\delta, \gamma) \subset D(\delta', \gamma]$

$$a \rightarrow \mathbf{k}(a).$$

Inclusion $[(0)^0, \gamma] \subset [(0)^0, \gamma']$ define a homomorphism

$$\mathbb{C}[(0)^0, \gamma'] \rightarrow \mathbb{C}[(0)^0, \gamma]$$

which is the identity on $\lambda^\alpha, \alpha \in [(0)^0, \gamma]$ and zero on $\lambda^\alpha, \alpha \in [(0)^0, \gamma'] \setminus [(0)^0, \gamma]$. I use it to define the maps

$$\text{pr} : S_{\mathbf{a}}[\delta, \gamma'] \rightarrow S_{\mathbf{a}}[\delta, \gamma]. \quad (4.76)$$

Note that maps i and pr commute and I can define the double limit

$$S_{\mathbf{a}}[\infty, \infty] = \lim_{\delta \rightarrow} \lim_{\gamma \leftarrow} S_{\mathbf{a}}[\delta, \gamma]. \quad (4.77)$$

$S_{\mathbf{a}}[\infty, \infty]$ is a module over $D(\infty, \infty)$.

Remark 4.78 The modules

$$S_{\mathbf{b}}[\delta, \gamma] := \mathbb{C}[\delta, (1)^{-1}] \otimes \mathbb{C}_-[(0)^0, \gamma] \otimes \mathbb{C}_+[(0)^0, \gamma]^{-1} \quad (4.79)$$

and $S_{\mathbf{a}'}[\delta, \gamma]$ form a bidirect system similar to (4.77). To define the structure involved in its definition, we have to conjugate all the constructions from Definition 4.72 with the automorphism $\tau\sigma$ (2.9,2.11) in case of $S_{\mathbf{b}}$ and with σ in case of $S_{\mathbf{a}'}$. In particular the pair $M_2^- \subset M_1^+$, that appear in Definition 4.72, has to be replaced by $M_2^+ \subset M_1^-$.

Finally

$$\begin{aligned} S_{\mathbf{f}}[\mathbf{K}] &:= \mathbb{C}_-[\mathbf{K}^{\leq -1}] \otimes \mathbb{C}_+[\mathbf{K}^{\leq -1}]^{-1} \otimes \mathbb{C}[\mathbf{K}^{\geq 0}], \\ S_{\mathbf{f}'}[\mathbf{K}] &:= \mathbb{C}[\mathbf{K}^{\leq -1}] \otimes \mathbb{C}_-[\mathbf{K}^{\geq 0}] \otimes \mathbb{C}_+[\mathbf{K}^{\geq 0}]^{-1}. \end{aligned} \quad (4.80)$$

There is an obvious map

$$S_{\mathbf{f}}[\delta, \gamma] \otimes S_{\mathbf{f}'}[\delta, \gamma] \rightarrow T_{\mathbf{m}}[\delta, \gamma]$$

Modules $S_{\mathbf{f}}[\delta, \gamma], S_{\mathbf{f}'}[\delta, \gamma]$ forma bidirect system with structure maps the same as \mathbf{k} (Definition 4.72) and pr (4.76). See also Remark 4.78.

There is a straightforward generalization of the above constructions for semi-closed and open intervals $(\delta, \gamma], [\delta, \gamma), (\delta, \gamma)$, which I will use freely.

4.2.2 Computation of $H_c^i(A)$ with complexes $T_{\bullet}(\mathbf{c})$ and $S_{\bullet}(\mathbf{c})$

I will implement the idea of Proposition 3.30 to use Tor-functors and modules from Section 4.2.1 for computation of $H_{\mathbf{a}}^i(A)$ for the algebra A based on the interval $[\delta, \delta']$.

Definition 4.81 With the help of the minimal P -resolution $F_{\bullet}(A)$ (3.31), I define the complexes

$$T_{\bullet}(\mathbf{c}) := F_{\bullet} \otimes_P T_{\mathbf{c}} \quad S_{\bullet}(\mathbf{c}) = F_{\bullet} \otimes_P S_{\mathbf{c}}$$

$\mathbf{c} = (0), \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{p}, \mathbf{f}, \mathbf{f}'$ or \mathbf{m} . See (4.54), (4.70), (4.79) and for definition of T and S .

Before I formulate precise relation of the cohomology $H_i(T(\mathfrak{c}))$ and $H_i(S(\mathfrak{c}))$ to $H_c^i(A)$, I will exhibit a system of parameters in \mathfrak{c} , which will justify construction of $S_\bullet(\mathfrak{c})$.

Recall that a sequence of elements $\{x_1, \dots, x_n\}$ in the ideal $\mathfrak{c} \subset R$ is a *system of parameters* if $\text{Rad}(x_1, \dots, x_n) = \mathfrak{c}$

Proposition 4.82 *Let \mathfrak{z} be an ideal of $A[\gamma, \alpha]$ generated by*

$$\text{Reg}([\gamma, \alpha] \setminus [\beta, \alpha]), \beta \leq \alpha \quad (4.83)$$

See (4.9) for definition of Reg . \mathfrak{z} contains in the ideal $\mathfrak{c} = \mathfrak{i}([\gamma, \alpha] \setminus [\beta, \alpha]) \subset A[\gamma, \alpha]$. There is also an ideal $\mathfrak{h} = \text{Reg}([\gamma, \alpha]^{\leq -1})$, which contains in $\mathfrak{f} = \mathfrak{i}([\gamma, \alpha]^{\leq -1})$. Then $\text{Rad } \mathfrak{z} = \mathfrak{c}$, $\text{Rad } \mathfrak{h} = \mathfrak{f}$.

Proof. I set $k = \min_{[\gamma, \alpha] \setminus [\beta, \alpha]} \rho(\gamma)$. Denote by \mathfrak{z}' the preimage of \mathfrak{z} in $\mathbb{C}[\gamma, \alpha]$. \mathfrak{z}' is generated by (λ^l) and the defining relations (2.2). Consider a filtration $\mathfrak{z}'_0 \subset \mathfrak{z}'_1 \subset \dots \subset \mathfrak{z}'$, where \mathfrak{z}'_i is generated by $\lambda^k, \dots, \lambda^{k+i}$ and (2.2).

Let \mathfrak{c}' be the preimage of \mathfrak{c} in $\mathbb{C}[\gamma, \alpha]$. Define the filtration $\mathfrak{c}'_0 \subset \mathfrak{c}'_1 \subset \dots \subset \mathfrak{c}'$. \mathfrak{c}'_i is generated by

$$G_i = \{\lambda^\alpha | \alpha \in [\gamma, \alpha] \setminus [\beta, \alpha], k \leq \rho(\alpha) \leq k + i\}.$$

For the proof it suffice to show that $\text{Rad } \mathfrak{z}' = \mathfrak{c}'$. Note that $\text{Rad } \mathfrak{c}' = \mathfrak{c}'$ because $P/\mathfrak{c}' = A[\gamma, \alpha]/\mathfrak{c} = A[\beta, \alpha]$ has no zero divisors [36].

The proposition will follow if I prove that $\text{Rad } \mathfrak{z}'_i \supset G_i$ for all i . Note that if $G_{i+1} \setminus G_i$ consists of one element, then this element is λ^{k+i} . This is true, for example, when $i = 0$. If, under assumption $|G_{i+1} \setminus G_i| = 1$, I have already proven that $\text{Rad } \mathfrak{z}'_i \supset G_i$, then $\text{Rad } \mathfrak{z}'_{i+1} \supset G_{i+1}$ will follow trivially. The remaining nontrivial case is $|G_{i+1} \setminus G_i| = 2$. Note that in this case there is one relation in (2.2) that contains a clutter $\lambda^\delta \lambda^{\delta'}$ with $\lambda^\delta, \lambda^{\delta'} \in G_{i+1} \setminus G_i$. By using straightening relations and working under inductive assumptions, I see that all other terms in this relation contain in $\text{Rad } \mathfrak{z}'_i$. By using λ^{k+i} for the Gröbner reductions, I transform the cluttering monomial $\lambda^\delta \lambda^{\delta'}$ to the forms $(\lambda^\delta)^2$ or $(\lambda^{\delta'})^2$. This implies that $\lambda^\delta, \lambda^{\delta'} \in \text{Rad } \mathfrak{z}'_{i+1}$.

The proof of the second equality is similar. $A[\gamma, \alpha]/\mathfrak{f} = A[\gamma, \alpha]^{\geq 0}$. The algebra $A[\gamma, \alpha]^{\geq 0}$ inherits standard basis from $A[\gamma, \alpha]$. By using this basis I construct inclusion $A[\gamma, \alpha]^{\geq 0} \rightarrow A[(0)^0, \alpha] + A[(3)^{-1}, \alpha]$. The summand have no zero devisors [36] thus the radical of $A[\gamma, \alpha]^{\geq 0}$ is trivial and $\mathfrak{f} = \text{Rad } \mathfrak{f}$. The remainder of the proof is similar to the previous case and omitted. ■

The following proposition explains me how to manipulate with the ideal in local cohomology group without changing the content of the cohomology.

Proposition 4.84 ([29], Proposition 7.2) Let M be an R -module. If radicals of two ideals $\mathfrak{a}, \mathfrak{b} \subset R$ coincide $\text{Rad } \mathfrak{a} = \text{Rad } \mathfrak{b} = \mathfrak{c}$, then

$$H_{\mathfrak{a}}^i(M) = H_{\mathfrak{b}}^i(M) = H_{\mathfrak{c}}^i(M).$$

Here is an example of how this theorem can be used.

Proposition 4.85 Let \mathfrak{c} be $\mathfrak{a}, \mathfrak{a}', \mathfrak{b}, \mathfrak{f}, \mathfrak{f}'$ or \mathfrak{m} .

1. Let $T_{\mathfrak{c}}$ be as in (4.54) . Then

$$H_{\mathfrak{c}}^i(A) \cong \text{Tor}_{s(\mathfrak{c})-i}^P(A, T_{\mathfrak{c}}), \quad \text{Tor}_i^P(A, T_{\mathfrak{c}}) = H_i(T(\mathfrak{c})) \quad (4.86)$$

where

$$s(\mathfrak{c}) = |\mathbf{N}(\mathfrak{c})|.$$

The isomorphisms are compatible with the Aut action.

2. Let $S_{\mathfrak{c}}$ be as in (4.70), (4.79), or (4.80). Then

$$H_{\mathfrak{c}}^i(A) \cong \text{Tor}_{s'(\mathfrak{c})-i}^P(A, S_{\mathfrak{c}}), \quad \text{Tor}_i^P(A, S_{\mathfrak{c}}) = H_i(S(\mathfrak{c}))$$

where

$$s'(\mathfrak{c}) = \begin{cases} 3 - \rho(\delta) & \text{if } \mathfrak{c} = \mathfrak{a} \\ \rho(\delta') - 8 & \text{if } \mathfrak{c} = \mathfrak{a}' \\ \rho(\delta') - 2 & \text{if } \mathfrak{c} = \mathfrak{b} \\ -\rho(\delta) & \text{if } \mathfrak{c} = \mathfrak{f} \\ \rho(\delta') + 1 & \text{if } \mathfrak{c} = \mathfrak{f}' \end{cases} \quad (4.87)$$

Proof. The proof of the first item follow from Proposition 3.30. The functions s give the number of generators of the ideal.

I prove the second item only for \mathfrak{a} .

Lemma 4.88 $H_{\mathfrak{a}}^i(A) = H_{(\bar{\lambda})}^i(A)$, where $\bar{\lambda} = \text{Reg}[\delta, (1)^{-1}]$, (see (4.9) for the definition of Reg).

Proof. The ideal \mathfrak{a} contains a system of parameters $(\bar{\lambda}) = \text{Reg}[\delta, (1)^{-1}]$ (see (4.83) and Proposition 4.82).

By Proposition 4.84 $H_{\mathfrak{a}}^i(A) = H_{(\bar{\lambda})}^i(A)$. ■

To finish the proof, it remains to apply Proposition 3.30. Note that $s'(\mathfrak{a}) = |\text{Reg}[\delta, (1)^{-1}]|$. ■

4.2.3 The *-duality pairing

In this section, I will study the pairing between the groups $H_{\mathfrak{a}}^i(A)$ and $H_{\mathfrak{b}}^j(A)$. It is a composition of the multiplication \mathfrak{m} (4.3), isomorphism (4.12) and the residue map res_A (4.15):

$$(a, b) := \text{res}_A(\mathfrak{m}(a, b)). \quad (4.89)$$

I will also discuss the pairing (a, b) between the groups $H_{\mathfrak{f}}^i(A)$ and $H_{\mathfrak{f}'}^j(A)$ defined by composition of (4.4) and res_A .

Proposition 4.90 *1. The bilinear form (a, b) (4.89) defines a perfect pairing between $H_{\mathfrak{a}}^i(A)$ and $H_{\mathfrak{b}}^j(A)$ of degree $\text{rk}(\delta, \delta') + 1$. The Aut-weight of the pairing is equal to*

$$(a[\delta, \delta'], u[\delta, \delta'], r[\delta, \delta']). \quad (\text{For definition see 4.45}). \quad (4.91)$$

2. (a, b) defines a perfect pairing between $H_{\mathfrak{f}}^i(A)$ and $H_{\mathfrak{f}'}^j(A)$ of degree $\text{rk}(\delta, \delta') + 1$. It has weight (4.91).

3. Suppose that the interval $[\delta, \delta'] = [(0)^N, (1)^{N'}]$. By Remark 4.2 the groups $H_{\mathfrak{a}}^i[(0)^N, (1)^{N'}]$, $H_{\mathfrak{b}}^i[(0)^N, (1)^{N'}]$ are equipped with $\text{Spin}(10)$ -action. The pairing is compatible with this action.

Proof.

By using Proposition 4.85, I identify local cohomology with the Tor groups, which I compute with the complexes $T_{\bullet}(\mathfrak{a})$ and $T_{\bullet}(\mathfrak{b})$.

By Proposition 3.43, the lift of the map \mathfrak{m} (4.3) on the level of chains $T_i(\mathfrak{a})$ and $T_i(\mathfrak{b})$ is the product

$$V_i \otimes T_{\mathfrak{a}} \otimes V_j \otimes T_{\mathfrak{b}} \rightarrow V_{i+j} \otimes T_{\mathfrak{m}} \quad (4.92)$$

of maps (3.42) and (4.64).

Lemma 4.93 *There are nondegenerate pairings*

$$T_i(\mathfrak{a}) \otimes T_{d-i}(\mathfrak{b}) \rightarrow \mathbb{C}$$

$$T_i((0)) \otimes T_{d-i}(\mathfrak{m}) \rightarrow \mathbb{C}$$

$$d := s(\mathfrak{m}) - s'(\mathfrak{m}).$$

Proof. The analogue of (4.92) for $T_{\bullet}((0))$ and $T_{\bullet}(\mathfrak{m})$ is

$$V_i \otimes T_{(0)} \otimes V_j \otimes T_{\mathfrak{m}} \rightarrow V_{i+j} \otimes T_{\mathfrak{m}}. \quad (4.94)$$

The pairing between the pair $T_\bullet(\mathfrak{a})$ and $T_\bullet(\mathfrak{b})$ is the composition of (4.92) with

$$V_d \otimes T_{\mathfrak{m}} \xrightarrow{\mu \otimes \text{res}_T} \mathbb{C}. \quad (4.95)$$

For the pair $T_\bullet((0))$ and $T_\bullet(\mathfrak{m})$ I have to compose (4.94) with (4.95). The map res_T is defined in (4.65) and μ in (3.41). The resulting pairings are the tensor product of two nondegenerate pairings (see 3.42 and Proposition 4.67).

■

The map (4.95) defined on cohomology $H_d(T(\mathfrak{m}))$ is proportional to (4.15). Indeed, $\mu \otimes \text{res}_T$ is homogeneous. It is nonzero only on the graded component of $V_d \otimes T_{\mathfrak{m}}$ spanned, in notations of the proof of Proposition 4.18, by $\lambda^{-1} \otimes v$. By the same proposition, $\deg_{\mathbb{C}^\times} \lambda^{-1} \otimes v = -a$ -the grading of $A_0^* \subset H_{\mathfrak{m}}^{\text{rk}(\delta, \delta') + 1}(A)$.

The groups Aut acts on A_0^* through the character $a_A(t, q, z)$ (4.16). By Proposition 4.19, $a_A(t, q, z) = \chi^{-1}$. Exponents of the monomial in t, q, z that define χ are tabulated in Corollary 4.44. From it I deduce the weight of res_A , which is equal to the weight of the pairing.

The proof of the second statement is similar and I omit it.

To prove the third statement I observe that by Lemma 3.34 the maps (3.42) are $\text{Spin}(10)$ -equivariant (I don't have to work with minimal resolutions). The map (4.64) is $\text{Spin}(10)$ -equivariant by construction. Thus the map $\mathfrak{m}(a, b)$ is $\text{Spin}(10)$ -equivariant on the level of cohomology. The group $\text{Spin}(10)$ is simple. It must act trivially on $A_0^* \subset H_{\mathfrak{m}}^{8(N' - N) + 11}[(0)^N, (1)^{N'}]$. Hence res is $\text{Spin}(10)$ -equivariant map. From this I conclude that $\text{resm}(a, b)$ is $\text{Spin}(10)$ -equivariant pairing. ■

Remark 4.96 The cohomology pairing (4.89) and the map $\text{res}_{A[\delta, \delta']}$ transform under Aut by the character $\chi[\delta, \delta']$ (see 4.45). It is convenient to twist the action of Aut on $H_{\mathfrak{a}}^i(A)$ by the character $\chi^{-1}[\delta, (1)^{-1}]$, $H_{\mathfrak{b}}^i(A)$ by $\chi^{-1}[(0)^0, \beta]$, $H_{\mathfrak{i}}^i(A)$ by $\chi^{-1}[\delta, \delta']^{\leq 1}$, $H_{\mathfrak{i}'}^i(A)$ by $\chi^{-1}[\delta, \delta']^{\geq 0}$, and $H_{\mathfrak{m}}^i(A)$ by $\chi^{-1}[\delta, \delta']$. Then according to (4.46) after the twist the Aut -weight of the pairing (4.3) is $t^{-4}q^2$. The pairing (4.4) is equivariant with respect to the twisted action. I will also refer to this twist as to *renormalized action* of Aut . The grading by weight spaces of the renormalized action will be called *the renormalized grading*.

4.2.4 The four-term complex $\text{Fock}_{\bullet}^{\mathfrak{a}}[\delta, \delta']$

In this section, I will define a very compact four-term complex $\text{Fock}_{\bullet}^{\mathfrak{a}}$ that computes $H_{\mathfrak{a}}^i(A)$ for algebra A based on an interval $[\delta, \delta']$. Let us fix some notations and recall the old (4.68). Denote the subset of $\hat{\mathbb{E}} \hat{\mathbb{E}}^{\leq 2} \cap \hat{\mathbb{E}}^{\geq 0}$ by A .

$$A_1 \cup A_2 := \{(0)^0, (12)^0, (13)^0\} \cup \{(3)^{-1}, (2)^{-1}, (1)^{-1}\} = A$$

$$A^c := \begin{cases} A_2 & \text{if } c = a \\ A_1 & \text{if } c = b \end{cases}$$

Introduce a tensor factorization of $P = \mathbb{C}[\delta, \delta'] = R^c \otimes Q^c, c = a, b :$

$$R^a := \mathbb{C}[[\delta, \delta'] \setminus A] \otimes \mathbb{C}[A_1], Q^a := \mathbb{C}[A^a], \quad R^b := \mathbb{C}[[\delta, \delta'] \setminus A] \otimes \mathbb{C}[A_2], Q^b := \mathbb{C}[A^b].$$

There will be two modifications of $\text{Fock}_\bullet^c[\delta, \delta']$. All of them are Koszul complexes based on Q^c -modules I_c :

$$I_c := S_c \otimes_{R^c} A. \quad (4.97)$$

The generators $\lambda^\alpha, \alpha \in A^c$ act on I_c by multiplication on $\underline{\lambda}^\alpha$ (see (A.14) for notations).

In the next proposition, I show that the four-term Koszul complexes

$$\text{Fock}_\bullet^c := B_\bullet(I_c, \underline{\lambda}^\alpha | \alpha \in A^c) \quad (4.98)$$

compute $H_c^i(A)$.

Proposition 4.99 *There is an isomorphism*

$$H_c^{s'(c)-i}[\delta, \delta'] \cong \text{Tor}_i^P(A[\delta, \delta'], S_c[\delta, \delta']) \cong H_i(\text{Fock}_\bullet^c[\delta, \delta']) \quad (4.100)$$

$s'(c), c = a, b$ (see (4.87) for the definition of s').

Proof. I prove the statement only for $c = a$. By Proposition 4.85, $H_a^{-\rho(\delta)+3-i} \cong \text{Tor}_i^P(A, S_a)$. The constant $-\rho(\delta)+3$ is the number of elements in the sequence $\text{Reg}[\delta, (1)^{-1}]$. For computation of Tor I use the Koszul resolution

$$B_\bullet(A \otimes P, \{\underline{\lambda}^\alpha\}, \alpha \in [\delta, \delta']). \quad (4.101)$$

It immediately implies that the cohomology of

$$B_\bullet(A \otimes S_a, \{\underline{\lambda}^\alpha\}) \quad (4.102)$$

computes $\text{Tor}_i^P(A, S_a)$.

My plan is to apply construction (A.12) to (4.102):

$$\begin{aligned} \partial_1 a &= \sum_{\alpha \in [\delta, \delta'] \setminus A} \underline{\lambda}^\alpha \frac{\partial a}{\partial \theta^\alpha} + \sum_{\alpha \in A_1} \underline{\lambda}^\alpha \frac{\partial a}{\partial \theta^\alpha} \\ \partial_2 a &= \sum_{\alpha \in A_2} \underline{\lambda}^\alpha \frac{\partial a}{\partial \theta^\alpha}. \end{aligned}$$

See (A.14) for notations. The proof is based on consideration of the spectral sequence of the bicomplex $B_{\bullet, \bullet}$.

Lemma 4.103 *The cohomology of $(B_{\bullet,j}, \partial_1)$ is the first page $E_{i,j}^1$ of the spectral sequence of the bicomplex. I claim that $E_{i,j}^1 = 0, i > 0$. $E_{0,j}^1 = \text{Fock}_j^{\mathfrak{a}}$.*

Proof. I will use the third item of Lemma A.13. I have the following identifications $R^{\mathfrak{a}} = H = H_1 \otimes H_2$,

$$\begin{aligned} H_1 &= \mathbb{C}_-[[\delta, \delta']^{\leq -1}] \otimes \mathbb{C}[A_1] \otimes \mathbb{C}_-[[\delta, \delta']^{\geq 3}], \\ H_2 &= \mathbb{C}_+[[\delta, \delta']^{\leq -1}] \otimes \mathbb{C}_+[[\delta, \delta']^{\geq 3}] = H_2' \otimes H_2''. \end{aligned}$$

$$S_{\mathfrak{a}} \cong H_1 \otimes H_2'^{-1} \otimes H_2'' \otimes \mathbb{C}[A_2]^{-1}. \quad (4.104)$$

$S_{\mathfrak{a}}$ is by construction a free H_1 -module. Lemma 4.13 implies that A is a free H_2 module. Conditions of Lemma A.13 are satisfied. I conclude that $E_{i,j}^1 = 0, i \geq 1$. Equality $E_{0,j}^1 = \text{Fock}_j^{\mathfrak{a}}$ trivially verifies.

■

This finishes the proof.

■

The next proposition gives an interpretation to space of chains of $\text{Fock}_i^{\mathfrak{c}}$ in terms of local cohomology.

Proposition 4.105 *There is an isomorphism*

$$I_{\mathfrak{a}} \cong H_{\mathfrak{f}}^{-\rho(\delta)}(A) \otimes \mathbb{C}[A^{\mathfrak{a}}]^{-1}, \quad I_{\mathfrak{b}} \cong H_{\mathfrak{f}}^{\rho(\delta')+1}(A) \otimes \mathbb{C}[A^{\mathfrak{b}}]^{-1} \quad (4.106)$$

Proof. I prove only for \mathfrak{a} . I denote $H_1 \otimes H_2'^{-1} \otimes H_2''$ in (4.104) by X . The tensor product A and X over $R^{\mathfrak{a}}$ coincides with the tensor product over $\mathbb{C}[\delta, \delta']$ of A and $S_{\mathfrak{f}}[\delta, \delta']$ (4.80). By Proposition 3.30 there is an isomorphism

$$A[\delta, \delta'] \otimes_P S_{\mathfrak{f}}[\delta, \delta'] = H_{\mathfrak{f}}^{-\rho(\delta)}(A), \quad (4.107)$$

where \mathfrak{f} is generated by the regular sequence $\text{Reg}[\delta, \delta']^{\leq -1}$. The proof follows from Proposition 4.82, Proposition 4.84. ■

Here is an immediate corollary of the proof.

Corollary 4.108

$$\begin{aligned} H_{\mathfrak{f}}^i(A) &\neq 0 \text{ only for } i = -\rho(\delta), \\ H_{\mathfrak{f}}^i(A) &\neq 0 \text{ only for } i = \rho(\delta') + 1. \end{aligned}$$

The fact that local cohomology $H_{\mathfrak{a}}^i(A)$ are nontrivial only for four consecutive values of the cohomological index is established in the following proposition.

Proposition 4.109

$$H_{\mathfrak{a}}^i(A) \neq 0 \text{ only for } -\rho(\delta) \leq i \leq -\rho(\delta) + 3,$$

$$H_{\mathfrak{b}}^i(A) \neq 0 \text{ only for } \rho(\delta') - 2 \leq i \leq \rho(\delta') + 1.$$

Proof. The statement is a corollary of Proposition 4.99. ■

4.2.5 Computations of the virtual character of $H_{\mathfrak{a}}^i(A)$

Fix an algebra A based on the interval $[\delta, \delta']$. The plan for this section is to derive functional equations satisfied by the virtual character

$$Z_{\mathfrak{c}}^{bare}(t, q, z) := \sum_{i \geq 0} (-1)^i \chi_{H_{\mathfrak{c}}^i(A)}(t, q, z), \mathfrak{c} = \mathfrak{a}, \mathfrak{b}$$

of Aut (1.20). The character $\chi_{H_{\mathfrak{c}}^i(A)}(t, q, z)$ is a generating function of dimensions of weight subspaces $\sum_{j,u,r} \dim H_{\mathfrak{c}}^{i,j,r}(A) t^j q^u z^r$. By using sets $N(\mathfrak{c})$ (4.51) and functions (4.45), I define

$$a(\mathfrak{c}) := a(N(\mathfrak{c})), u(\mathfrak{c}) := u(N(\mathfrak{c})), r(\mathfrak{c}) := r(N(\mathfrak{c})).$$

Together with $Z_{\mathfrak{a}}^{bare}$ I also study its renormalized version

$$Z_{\mathfrak{a}} := (-1)^{s'(\mathfrak{a})} t^{a(\mathfrak{a})} q^{u(\mathfrak{a})} z^{r(\mathfrak{a})} Z_{\mathfrak{a}}^{bare}. \quad (4.110)$$

The function s' is as in (4.87). This virtual character corresponds to the twisted action described in Remark 4.96.

Here is the functional equation for Z in the most basic form.

Proposition 4.111 $Z_{\mathfrak{a}}^{bare}$ and $Z_{\mathfrak{a}}$ satisfy

$$\begin{aligned} Z_{\mathfrak{a}}^{bare}(t, q, z) &= \\ &= (-1)^{s'(\mathfrak{m})} t^{-a(\mathfrak{m})} q^{-u(\mathfrak{m})} z^{-r(\mathfrak{m})} Z_{\mathfrak{b}}^{bare}(t^{-1}, q^{-1}, z^{-1}) \end{aligned} \quad (4.112)$$

and

$$Z_{\mathfrak{b}}^{bare}[\delta, \delta'](t, q, z) = Z_{\mathfrak{a}}^{bare}[\tau^{-1}\sigma(\delta'), \tau^{-1}\sigma(\delta)](q^{-1}t, q^{-1}, Sz). \quad (4.113)$$

Proof. The formula (4.112) follows from Proposition 4.90 and formula (2.45).

The composition τ^{-1} and σ (4.5) induce an isomorphism

$$\tau^{-1}\sigma : H_{\mathfrak{b}}^i[\delta, \delta'] \rightarrow H_{\mathfrak{a}}^i[\tau^{-1}\sigma(\delta'), \tau^{-1}\sigma(\delta)]. \quad (4.114)$$

Here are the rules of commuting elements of Aut and σ and τ .

$$\begin{aligned}
\sigma((t, q, z)a) &= (t, q^{-1}, Sz)\sigma(a) \\
\tau((t, q, z)a) &= (tq^{-1}, q, z)T(a) \\
(t, q, z) &\in Aut, a \in A \\
S \in D_5 &\text{ is a certain automorphism of } \tilde{\mathbf{T}}^5 \\
&\text{(see Appendix A in [36] for details).}
\end{aligned} \tag{4.115}$$

These facts imply the formula (4.113). ■

Here is an alternative way to write the same set of equations.

Proposition 4.116 *The virtual characters $Z_{\mathbf{a}}^{bare}$ and $Z_{\mathbf{a}}$ are solutions of the functional equations*

$$\begin{aligned}
Z_{\mathbf{a}}^{bare}[\delta, \delta'](t, q, z) &= \\
&= (-1)^{s'(\mathbf{m})} t^{-a(\mathbf{m})} q^{-u(\mathbf{m})} z^{-r(\mathbf{m})} Z_{\mathbf{a}}^{bare}[\tau^{-1}\sigma(\delta'), \tau^{-1}\sigma(\delta)](qt^{-1}, q, Sz^{-1})
\end{aligned} \tag{4.117}$$

and

$$Z_{\mathbf{a}}[\delta, \delta'](t, q, z) = -t^{-4}q^2 Z_{\mathbf{a}}[\tau^{-1}\sigma(\delta'), \tau^{-1}\sigma(\delta)](qt^{-1}, q, Sz^{-1}). \tag{4.118}$$

Proof. The first formula is a direct corollary of Proposition 4.111.

Derivation of the second formula uses formula (4.117), items (2, 4) from Proposition 4.44 and equation (2.45), which sum up to

$$\begin{aligned}
a(\mathbf{m}) &= a(\mathbf{a}) + a(\mathbf{b}) + 4, \\
u(\mathbf{m}) &= u(\mathbf{a}) + u(\mathbf{b}) - 2, \\
r(\mathbf{m}) &= r(\mathbf{a}) + r(\mathbf{b}).
\end{aligned}$$

■

Equation (4.118) simplifies if I set $[\delta, \delta'] = [(0)^{-N-1}, (1)^N]$. Denote $Z_{\mathbf{a}}[(0)^{-N-1}, (1)^N]$ by $Z[N]$. Then (4.118) becomes (?). Operator S disappears in Sz^{-1} because representation of $\tilde{\mathbf{T}}^5$ in $H_{\mathbf{a}}^i(A[(0)^{-N-1}, (1)^N])$ is a restriction of $\text{Spin}(10, \mathbb{C})$ -representation and $S(z) = SzS^{-1}$, $S \in \text{Spin}(10, \mathbb{C})$ is an inner automorphism.

An explicit formula for $Z_{\mathbf{a}}$ Isomorphism (4.86) can be used for derivation of an explicit formula for $Z_{\mathbf{a}}$:

$$Z_{\mathbf{c}}^{bare}(t, q, z) = (-1)^{|\delta, (1)^{-1}|} \sum (-1)^i \text{Tor}_i^P(A, T_{\mathbf{c}})(t, q, z).$$

For computation of Tor groups I can use resolution (4.101). The corresponding complex that computes Tor groups is $\{A \otimes T_{\mathfrak{c}} \otimes \Lambda^i S_+\}$. I conclude that

$$Z_{\mathfrak{a}}^{bare}(t, q, z) = (-1)^{|\delta, (1)^{-1}|} A(t, q, z) T_{\mathfrak{a}}(t, q, z) \Lambda S_+(-t, q, z). \quad (4.119)$$

I computed $A(t, q, z)$ in [36].

$$\begin{aligned} T_{\mathfrak{a}}(t, q, z) &= \prod_{\alpha \in N(\mathfrak{a})} \frac{\hat{v}_{\alpha}^{-1}}{1 - \hat{v}_{\alpha}^{-1}} \prod_{\alpha \in E} \frac{1}{1 - \hat{v}_{\alpha}} \prod_{\alpha \in N(\mathfrak{a}')} \frac{1}{1 - \hat{v}_{\alpha}}, \\ \Lambda S_+(t, q, z) &= \prod_{\alpha \in N(\mathfrak{m})} (1 + \hat{v}_{\alpha}) \end{aligned}$$

(see notation (2.28)). One has to use caution with the formula 4.119. The problem is that the left-hand-side is defined as a rational function for any ideal $\mathfrak{c} = \mathfrak{i}[\delta, (1)^k]$ its components, specifically $T_{\mathfrak{i}}(t, q, z)$, makes sense as a formal power series in q only when $k = -1$. In this case $T_{\mathfrak{a}}(t, q, z) \Lambda S_+(-t, q, z) = 1$ in the ring of formal power series and

$$Z_{\mathfrak{a}}^{bare}(t, q, z) = (-1)^{|\delta, (1)^{-1}|} A(t, q, z)$$

The formula for $A_N^{N'} := A[(0)^N, (1)^{N'}](t, q, 1)$, $B_N^{N'} := A[(0)^N, (0)^{N'}](t, q, 1)$, $N < 0 < N'$ can be simplified (see [36] for details). In [36] I established that

$$\begin{aligned} B_0^1 &= \frac{1 + 3t + t^2}{(1-t)^8(1-qt)}, \quad A_0^0 = \frac{1 + 5t + 5t^2 + t^3}{(1-t)^{11}}, \\ K(t, q) &= \begin{pmatrix} \frac{t(t^2+3t+1)}{(t-1)^9(qt-1)} & \frac{(t^2+3t+1)(t^3+q^2)-5q(t+1)t^2}{q^2(t-1)^9(qt-1)} \\ \frac{t(t+1)(t^2+4t+1)}{(t-1)^{10}} & \frac{(t^3+5t^2+5t+1)(t^3+q^2)-q(5t^2+14t+5)t^2}{q^2(t-1)^{10}} \end{pmatrix} \quad (4.120) \\ \begin{pmatrix} B_0^{r+1} \\ A_0^r \end{pmatrix} &= K(q^r t, q) \cdots K(qt, q) \begin{pmatrix} B_0^1 \\ A_0^0 \end{pmatrix}, \\ A_N^{N'}(t, q) &= A_0^{N'-N}(tq^N, q). \end{aligned}$$

The partition function

$$Z_{\mathfrak{a}}^{N'} = A_N^{N'}(t, q) \chi[(0)^N, (1)^{-1}] = A_N^{N'}(t, q) t^{4-4N} q^{-2+4N-2N^2}, N < 0 \quad (4.121)$$

The character $\chi[(0)^N, (1)^{-1}]$ is taken from Corollary 4.49.

Here are some terms of the q -expansion of $Z_{\mathfrak{a}} = Z_{-2}q^{-2} + Z_{-1}q^{-1} + Z_0q^0 + Z_1q^1 + \dots$

$$\begin{aligned}
& \frac{t^4(1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8)}{q^2(1-t)^{16}} + \\
& + \frac{t^4(46 - 144t + 116t^2 + 16t^3 - 16t^5 - 116t^6 + 144t^7 - 46t^8)}{q(1-t)^{16}} + \\
& + \frac{1}{(1-t)^{16}}(-1 + 16t - 120t^2 + 576t^3 - 1003t^4 + 528t^5 - 214t^6 + 592t^7 \\
& - 592t^9 + 214t^{10} - 528t^{11} + 1003t^{12} - 576t^{13} + 120t^{14} - 16t^{15} + t^{16}) + \\
& + \frac{q(-16 + 210t - 1200t^2 + 3696t^3 - 4704t^4 + 2630t^5 - 3312t^6 + 3148t^7 + 1328t^8 - 1328t^{10} - 3148t^{11} + \dots)}{t(1-t)^{16}} + \dots
\end{aligned} \tag{4.122}$$

See Remark 5.20 for interpretation of the coefficients of this expansion.

4.3 $H_{\mathfrak{a}}^i[\delta, \delta']$ as a function of δ , and δ'

To define the limiting space of states $H_{\mathfrak{a}}^{i+\frac{\infty}{2}}(A)$, I have to glue together spaces $H_{\mathfrak{a}}^i[\delta, \delta']$. This is not an obvious procedure because that range $-\rho(\delta) \leq i \leq -\rho(\delta) + 3$, for which the groups are nontrivial, change with δ (Proposition 4.109). The space $H_{\mathfrak{a}}^{i+\frac{\infty}{2}}(A)$ will be a limit of bidirect system formed by $H_{\mathfrak{a}}^i[\delta, \delta']$. My plan for this section is to define the structure maps of this system and study their basic properties.

In this section all intervals $[\delta_i, \delta_j]$ satisfy $\delta_i \leq (0)^0, (1)^{-1} \leq \delta_j$.

Let us consider a sequence $\overline{\lambda} = \text{Reg}([\delta, (1)^{-1}])$ (4.9). By Lemma 4.88 and equation (A.4)

$$H_{\mathfrak{a}}^i[\delta, \delta'] = H_{(\overline{\lambda})}^i[\delta, \delta'] = \lim_{\overrightarrow{n}} H^i(K(A[\delta, \delta'], \overline{\lambda}^n)). \tag{4.123}$$

Results of Subsection 3 will enable me to define bidirect system of linear spaces $\{H^i(K(A[\delta, \delta'], \overline{\lambda}^n))\}$ with varying δ, δ', n . Let us see how this can be implemented.

In the following proposition I will suppress $\overline{\lambda}^n$ -dependence in cohomology of Koszul complexes, which will be denoted by $HK^i[\delta, \delta']$.

Proposition 4.124 *Fix $\delta_1 < \delta_2 < (0)^{-1}, (1)^{-1} < \delta_3 < \delta_4$.*

There is a commutative diagram

$$\begin{array}{ccc}
HK^i[\delta_2, \delta_4] & \xrightarrow{p'} & HK^i[\delta_2, \delta_3] \\
\downarrow \text{th}_{\mathfrak{q}} & & \downarrow \text{th}_{\mathfrak{q}'} \\
HK^{i+\text{codim}}[\delta_1, \delta_4] & \xrightarrow{p} & HK^{i+\text{codim}}[\delta_1, \delta_3].
\end{array} \tag{4.125}$$

The maps are induced from the commutative diagram of algebras

$$\begin{array}{ccc} A[\delta_2, \delta_4] & \xrightarrow{\mathbf{p}'} & A[\delta_2, \delta_3] \\ \uparrow \mathbf{q} & & \uparrow \mathbf{q}' \\ A[\delta_1, \delta_4] & \xrightarrow{\mathbf{p}} & A[\delta_1, \delta_3] \end{array} \quad (4.126)$$

and maps of resolutions $\mathbf{th}_q, \mathbf{th}_{q'}$ (3.5, Definition 4.32).

$$\text{codim} = \rho(\delta_2) - \rho(\delta_1) = \text{rk}(\delta_1, \delta_2).$$

Proof.

First, I assume that \mathbf{q} and \mathbf{q}' is defined by the formulas (4.25, 4.26). By Lemma 4.24, items 1, 2 and Proposition 3.9, \mathbf{th}_q is encoded by the class of the extension

$$\{0\} \rightarrow A[\delta_1, \delta_4] \xrightarrow{\lambda^{\delta_1} \times ?} A[\delta_1, \delta_4] \xrightarrow{\mathbf{p}} A(\delta_1, \delta_4) \rightarrow \{0\}. \quad (4.127)$$

$\lambda^{\delta_1} \times ?$ stands for operator of multiplication on λ^{δ_1} . In (4.125) \mathbf{th}_q is the boundary homomorphism in the Koszul cohomology corresponding to this extension. After applying homomorphisms

$$\mathbf{p} : A[\delta_1, \delta_4] \rightarrow A[\delta_1, \delta_3], \mathbf{p}' : A(\delta_1, \delta_4) \rightarrow A(\delta_1, \delta_3) \quad (4.128)$$

(4.127) transforms to

$$\{0\} \rightarrow A[\delta_1, \delta_3] \xrightarrow{\lambda^{\delta_1} \times ?} A[\delta_1, \delta_3] \rightarrow A(\delta_1, \delta_3) \rightarrow \{0\},$$

which implies commutativity of (4.125).

Let us do the case when \mathbf{q} and \mathbf{q}' belong to the class (4.27). By Lemma 4.24, item 2, conditions of Proposition 3.10 are satisfied. By this proposition, the maps $\mathbf{th}_q, \mathbf{th}_{q'}$ (3.5) are inclusions

$$\mathbf{w} : A[\delta_2, \delta_i] \xrightarrow{\lambda^{\delta_2} \times ?} A(\delta_1, \delta_i), \quad (4.129)$$

which are obviously compatible with the specialization (4.128). See Proposition 3.10 for details on \mathbf{w} . This implies commutativity of (4.125) in the case when maps \mathbf{q} and \mathbf{q}' are defined by (4.27).

By my assumptions, conditions of Lemma 4.29 for the pair $[\delta_2, \delta_4] \subset [\delta_1, \delta_4]$ are satisfied. It remains to apply my previous arguments to elementary homomorphisms (4.31) and use Proposition 3.7.

■

Remark 4.130 *The proof of the Proposition would have been easier, had I known that in the commutative diagram*

$$\begin{array}{ccc} x \in \text{Ext}^{\text{codim}}(A[\delta_2, \delta_3], A[\delta_1, \delta_4]) & \xrightarrow{\text{id} \otimes \mathbf{p}} & \text{Ext}^{\text{codim}}(A[\delta_2, \delta_3], A[\delta_1, \delta_3]) \ni \mathbf{th}_{q'} \\ \downarrow \mathbf{p}'^* \otimes \text{id} & & \downarrow \mathbf{p}'^* \otimes \text{id} \\ \mathbf{th}_q \in \text{Ext}^{\text{codim}}(A[\delta_2, \delta_4], A[\delta_1, \delta_4]) & \xrightarrow{\text{id} \otimes \mathbf{p}} & \text{Ext}^{\text{codim}}(A[\delta_2, \delta_4], A[\delta_1, \delta_3]) \ni y \end{array}$$

exist elements x, y such that

$$\text{th}_q = p'^* \otimes \text{id}(x) \text{ and } \text{th}_{q'} = \text{id} \otimes p(x).$$

Then

$$y = p'^* \otimes \text{id}(\text{th}_{q'}) = \text{id} \otimes p(\text{th}_q).$$

In my formulas $\text{codim} = \text{rk}(\delta_1, \delta_2)$. This would imply commutativity of (4.125) and (4.140).

Corollary 4.131 *Under assumptions of Proposition 4.124 there is a commutative diagram*

$$\begin{array}{ccc} H_a^i[\delta_2, \delta_4] & \xrightarrow{p'} & H_a^i[\delta_2, \delta_3] \\ \downarrow \text{th}_q & & \downarrow \text{th}_{q'} \\ H_a^{i+\text{codim}}[\delta_1, \delta_4] & \xrightarrow{p} & H_a^{i+\text{codim}}[\delta_1, \delta_3]. \end{array} \quad (4.132)$$

Remark 4.133 The formulas (3.28), (4.35), and (4.40) imply that arrows in (4.132) commute with the renormalized *Aut* action defined in Remark 4.96.

4.3.1 Compatibility of the *-duality pairing and th

If we wish to extend pairing which exists on $H_a^i[\delta, \delta']$ to $H_a^{i+\frac{\infty}{2}}$, we have to verify consistency of the system of maps *th* and the pairing. This is what is done in this section.

The pairing (4.89) is the composition of *m* and *res*. I check compatibility of *th* with *m* is the next proposition.

Proposition 4.134 *Fix $[\delta_2, \delta_4] \subset [\delta_1, \delta_4]$. There is a commutative diagram*

$$\begin{array}{ccc} H_a^i[\delta_2, \delta_4] \otimes H_b^j[\delta_1, \delta_4] & \xrightarrow{m \circ (\text{id} \otimes p)} & H_m^{i+j}[\delta_2, \delta_4] \\ \downarrow \text{th}_p \otimes \text{id} & & \downarrow \text{th}_p \\ H_a^{i+\text{codim}}[\delta_1, \delta_4] \otimes H_b^j[\delta_1, \delta_4] & \xrightarrow{m} & H_m^{i+j+\text{codim}}[\delta_1, \delta_4] \end{array}$$

where *p* is the map induced by the restriction homomorphism $A[\delta_1, \delta_4] \rightarrow A[\delta_2, \delta_4]$,

$\text{th}_p \in \text{Ext}_{\mathbb{C}[\delta_1, \delta_4]}^{\text{codim}}(A[\delta_2, \delta_4], A[\delta_1, \delta_4])$ is the corresponding Thom class (Definition 4.32), *m* is the product map (4.3).

Proof. By using Proposition 4.85, I identify local cohomology with the Tor groups. It suffice to prove commutativity of

$$\begin{array}{ccc} \text{Tor}_i^P(A[\delta_2, \delta_4], T_a) \otimes \text{Tor}_j^P(A[\delta_1, \delta_4], T_b) & \xrightarrow{m \circ (\text{id} \otimes p)} & \text{Tor}_{i+j}^P(A[\delta_2, \delta_4], T_m) \\ \downarrow \text{th}_p \otimes \text{id} & & \downarrow \text{th}_p \\ \text{Tor}_{i-\text{codim}}^P(A[\delta_1, \delta_4], T_a) \otimes \text{Tor}_j^P(A[\delta_1, \delta_4], T_b) & \xrightarrow{m} & \text{Tor}_{i+j-\text{codim}}^P(A[\delta_1, \delta_4], T_m) \end{array}$$

where $T_{\mathfrak{a}} = T_{\mathfrak{a}}[\delta_1, \delta_4]$, $T_{\mathfrak{b}} = T_{\mathfrak{b}}[\delta_1, \delta_4]$, $T_{\mathfrak{m}} = T_{\mathfrak{m}}[\delta_1, \delta_4]$ as in Section 4.2.1 and $P = \mathbb{C}[\delta_1, \delta_4]$. Note that by Proposition 3.30, the groups $\{\mathrm{Tor}_i^{\mathbb{C}[\delta_1, \delta_4]}(A[\delta_2, \delta_4], T_{\mathfrak{a}}[\delta_1, \delta_4])\}$ are equal to $\{\mathrm{Tor}_i^{\mathbb{C}[\delta_2, \delta_4]}(A[\delta_2, \delta_4], T_{\mathfrak{a}}[\delta_2, \delta_4])\}$ with some index shift.

As in the proof of Proposition 4.124, I first verify the statement for $\mathrm{th}_{\mathfrak{p}}$ which belongs to one of the groups (4.25, 4.26 or 4.27). $\mathrm{th}_{\mathfrak{p}}$ from (4.25, 4.26) are the boundary homomorphisms. By [13] Proposition 2.5 Chap XI, I have a commutative diagram that is related to the boundary homomorphism corresponding to the extension (4.127)

$$\begin{array}{ccc} \mathrm{Tor}_i^P(A[\delta_1, \delta_4], T_{\mathfrak{a}}) \otimes \mathrm{Tor}_j^P(A[\delta_1, \delta_4], T_{\mathfrak{b}}) & \longrightarrow & \mathrm{Tor}_{i+j}^{P \otimes P}(A[\delta_1, \delta_4] \otimes A[\delta_1, \delta_4], T_{\mathfrak{a}} \otimes T_{\mathfrak{b}}) \\ \downarrow \mathrm{th}_{\mathfrak{p}} \otimes \mathrm{id} & & \downarrow \tilde{\mathrm{th}}_{\mathfrak{p}} \\ \mathrm{Tor}_{i-1}^P(A[\delta_1, \delta_4], T_{\mathfrak{a}}) \otimes \mathrm{Tor}_j^P(A[\delta_1, \delta_4], T_{\mathfrak{b}}) & \longrightarrow & \mathrm{Tor}_{i+j-1}^{P \otimes P}(A[\delta_1, \delta_4] \otimes A[\delta_1, \delta_4], T_{\mathfrak{a}} \otimes T_{\mathfrak{b}}) \end{array} \quad (4.135)$$

$\tilde{\mathrm{th}}_{\mathfrak{p}}$ is boundary map corresponding to extension (4.127) tensored over \mathbb{C} on $A[\delta_1, \delta_4]$. Observe that for any homomorphism $C \rightarrow B$ of commutative algebras the natural map

$$\mathrm{Tor}_i^C(M, N) \rightarrow \mathrm{Tor}_i^B(M \otimes_C B, N \otimes_C B) \quad (4.136)$$

is compatible with the boundary homomorphisms (whenever $? \otimes_A B$ transforms an exact sequence to an exact sequence). I apply this comment to $C = P \otimes P$, $B = P$, and the right column of 4.135. This way obtain the claim for $\mathrm{th}_{\mathfrak{p}}$ from (4.25, 4.26). For \mathfrak{p} from (4.27) the proof is similar but relies only on the fact that (4.136) is a map of functors.

I finish the proof as Proposition 4.124 by applying Lemma 4.29 for the pair $[\delta_2, \delta_4] \subset [\delta_1, \delta_4]$, decomposing $\mathrm{th}_{\mathfrak{p}}$ into the Yoneda product of elementary $\mathrm{th}_{\mathfrak{p}_i}$ I already analyzed. By using associativity of the product, I prove by induction $\mathrm{th}_{\mathfrak{p}_n} \cdots \mathrm{th}_{\mathfrak{p}_{i+1}} \mathfrak{m}((\mathrm{th}_{\mathfrak{p}_i} \cdots \mathrm{th}_{\mathfrak{p}_1} a) \mathfrak{p}(b)) = \mathrm{th}_{\mathfrak{p}_n} \cdots \mathrm{th}_{\mathfrak{p}_{i+2}} \mathfrak{m}((\mathrm{th}_{\mathfrak{p}_{i+1}} \mathrm{th}_{\mathfrak{p}_i} \cdots \mathrm{th}_{\mathfrak{p}_1} a) \mathfrak{p}(b))$.

■

Proposition 4.137 *In the assumptions of Proposition 4.134, the map*

$$\mathrm{th}_{\mathfrak{p}} : H_{\mathfrak{m}}^{\mathrm{rk}(\delta_2, \delta_4)+1}[\delta_2, \delta_4] \rightarrow H_{\mathfrak{m}}^{\mathrm{rk}(\delta_1, \delta_4)+1}[\delta_1, \delta_4]$$

is injective.

Proof. I will use the complex $T_{\bullet}(\mathfrak{m})$ for computation of $H_{\mathfrak{m}}^i(A)$ (Proposition 4.85). The map $\mathrm{th}_{\mathfrak{p}} : T_i(\mathfrak{m})[\delta_2, \delta_4] \rightarrow T_{i-\mathrm{codim}}(\mathfrak{m})[\delta_1, \delta_4]$ is the \mathbb{C} -adjoint to the map $\mathfrak{p} : F_{\bullet}(A[\delta_1, \delta_4]) \rightarrow F_{\bullet}(A[\delta_2, \delta_4])$, which is understood in the sense of the graded duality relative to $\deg_{\mathbb{C}^{\times}}$ -grading. Then on the level of cohomology $\mathrm{th}_{\mathfrak{p}}$ is \mathbb{C} -adjoint to the onto map $\mathfrak{p} : A[\delta_1, \delta_4] \rightarrow A[\delta_2, \delta_4]$. The adjoint must be injective. ■

The central result of this section is the formula for the adjoint for th . It appears in the next proposition.

Proposition 4.138 *In the assumptions of Proposition 4.134 the pairing (4.89) satisfies*

$$(\text{th}_{\mathbf{p}}(a), b) = (a, \mathbf{p}(b))$$

where $a \in H_{\mathbf{a}}^i[\delta_2, \delta_4], b \in H_{\mathbf{b}}^j[\delta_1, \delta_4], i + j + \rho(\delta_2) - \rho(\delta_1) = \text{rk}(\delta_1, \delta_4) + 1$.

The weight of $\text{th}_{\mathbf{p}}$ with respect to the renormalized Aut action (see Remark 4.96) is equal to zero.

Proof. It follows from Propositions 4.134, 4.137 because

$$\begin{aligned} (\text{th}_{\mathbf{p}}(a), b) &= \text{res}_{A[\delta_1, \delta_4]} \circ \mathbf{m}(\text{th}_{\mathbf{p}}(a), b) = \text{res}_{A[\delta_1, \delta_4]} \circ \text{th}_{\mathbf{p}} \circ \mathbf{m}(a, \mathbf{p}(b)) = \\ &= \text{res}_{A[\delta_2, \delta_4]} \circ \mathbf{m}(a, \mathbf{p}(b)) = (a, \mathbf{p}(b)). \end{aligned}$$

By Propositions 3.16, 4.137, I can normalize $\text{res}_{A[\delta_2, \delta_4]}$ so that it is equal to $\text{res}_{A[\delta_1, \delta_4]} \circ \text{th}_{\mathbf{p}}$.

It is suffice to prove the second statement for elementary maps \mathbf{p}_i from Lemma 4.29. By (3.28) and (4.41, 4.42, 4.43) the renormalized weight of $\text{th}_{\mathbf{p}_i}$ is equal to zero. ■

The following statement is a modification of Corollary 4.138. It could be interpreted as the statement of compatibility of the bidirect system defined by (4.132) and the pairing.

Proposition 4.139 *The graded dual (with respect to Aut -weight subspaces decomposition) of the diagram (4.132) is isomorphic to*

$$\begin{array}{ccc} H_{\mathbf{b}}^{i'}[\delta_2, \delta_4] & \xrightarrow{\text{th}_{\mathbf{p}'}} & H_{\mathbf{b}}^{i' + \text{codim}'}[\delta_2, \delta_3] \\ \uparrow \mathbf{q} & & \uparrow \mathbf{q}' \\ H_{\mathbf{b}}^{i'}[\delta_1, \delta_4] & \xrightarrow{\text{th}_{\mathbf{p}}} & H_{\mathbf{b}}^{i' + \text{codim}'}[\delta_1, \delta_3]. \end{array} \quad (4.140)$$

The maps are induced from the commutative diagram of algebras (4.126)

$$\text{codim}' = \rho(\delta_4) - \rho(\delta_3), i + i' = \rho(\delta_4) - \rho(\delta_2).$$

Proof. I start the proof with a lemma.

Lemma 4.141 1. $H_{\mathbf{a}}^i(A)$ is a positive energy \mathbf{T} -space.

2. $H_{\mathbf{b}}^i(A)$ is a negative energy \mathbf{T} -space.

3. $H_{\mathbf{m}}^i[\delta, (1)^{-1}]$ is a positive energy \mathbf{T} -space with the lowest \mathbf{T} weight $u[\delta, (1)^{-1}]$ (4.45).

Proof. By Lemma 3.34, $F_i(A[\gamma, \gamma']) = V_i \otimes \mathbb{C}[\gamma, \gamma']$ where V_i are finite-dimensional \mathbf{T} -representations. By Remark 4.62), $T_{\mathfrak{a}}$ ($T_{\mathfrak{b}}$) is a positive (negative) energy space with respect to the $\deg_{\mathbf{T}}$ -grading. From this I conclude that $T_i(\mathfrak{a})$ and $H_{\mathfrak{a}}^i(A)$ are positive energy spaces. Spaces $T_i(\mathfrak{b})$ and $H_{\mathfrak{b}}^i(A)$ have negative energy.

The proof of the third item follows from Proposition 4.11 and Proposition 4.18. ■

I conclude that the pairing (4.89) splits into a sum of nondegenerate pairings between finite-dimensional weight spaces.

The statement follow from Corollary 4.131 and Corollary 4.138.

■

Commutation rules for th , σ and τ are given below.

Proposition 4.142 *Fix $\delta < \delta' < \gamma < \gamma'$.*

Let

$$\mathfrak{q} : A[\sigma(\gamma'), \sigma(\delta)] \rightarrow A[\sigma(\gamma'), \sigma(\delta)], \quad q = \sigma \circ p \circ \sigma$$

where $\mathfrak{p} : A[\delta, \gamma'] \rightarrow A[\delta', \gamma']$. Then

$$\sigma \circ \text{th}_{\mathfrak{p}} = \text{th}_{\mathfrak{q}} \circ \sigma.$$

The map τ (2.10) satisfies

$$\tau \circ \text{th}_{\mathfrak{p}} = \text{th}_{\mathfrak{p}} \circ \tau.$$

There are similar identities for projections $\mathfrak{p} : A[\delta, \gamma'] \rightarrow A[\delta, \gamma]$.

Proof. I leave the proof as an exercise. ■

5 Bounds on weights and on dimensions of weight spaces of

$$H_{\mathfrak{a}}^i[\delta, \delta']$$

The spaces of states $H_{\mathfrak{a}}^{i+\frac{\infty}{2}}$ which I will define in Section 6 carry the $\deg_{\mathbf{T}}$ -grading. The finite approximations $H_{\mathfrak{a}}^i[\delta, \delta']$ are positive energy \mathbf{T} -representations (Lemma 4.141). Does $H_{\mathfrak{a}}^{i+\frac{\infty}{2}}$ share this property? It should, if we hope that $H_{\mathfrak{a}}^{i+\frac{\infty}{2}}$ is a representation of the Virasoro algebra with weights bounded from below, as discussed in the introduction. In this section I will do all the technical work needed to answer this question.

Let A be the algebra based on $[\delta, \delta']$. I will recast, as usual, my local cohomology computations to computations of certain Tor-functors (Proposition 3.30). It will be convenient to do computations of Tor

groups by using a special non-minimal free resolution of A . Algebra A has Koszul property [36]. I will use a special P -resolutions of A which comes with any Koszul algebra. I will spend the next section reviewing this.

5.1 The quadratic dual

In this section, I will describe the Koszul resolution mentioned in the previous paragraph. Recall (see e.g. [40]) that an algebra B is *G-quadratic* if its defining ideal has a Gröbner basis of quadrics with respect to some system of generators and some term order. In [36] I showed that A is G-quadratic in the order (8.2).

The quadratic dual $A^!$ (see e.g. [39], p. 5 for the definition) has the following description. Let

$$T := \bigoplus_{n \geq 0} S_+[\delta, \delta']^{\otimes n}$$

be the free graded algebra on $S_+[\delta, \delta']$ (2.8) in degree minus one. I choose negative grading to make it compatible with Chevalley-Eilenberg construction.

Let $\Gamma(5)$ be the $\text{Spin}(10)$ -intertwiner $\Lambda^5 V \rightarrow \text{Sym}^2 \check{S}_+$ (see [48]). In the base $\{v_s\}, \{\theta_\beta\}$ it defines the tensor $\Gamma_{s_1, \dots, s_5}^{\alpha\beta}$. Define a set of relations

$$r_{s_1, \dots, s_5}^{l, l'} = \sum_{\alpha\beta \in E} \Gamma_{s_1, \dots, s_5}^{\alpha\beta} [\theta_{\alpha^l}, \theta_{\beta^{l'}}] \sim 0 \quad 1 \leq s_1 < \dots < s_5 \leq 10, \alpha^l, \beta^{l'} \in [\delta, \delta'] \quad (5.1)$$

$[\cdot, \cdot]$ stands for the graded commutator.

Here is the homological interpretation of $A^!$.

Proposition 5.2 [36] *The algebra $\text{Ext}_A(\mathbb{C}, \mathbb{C})$ is isomorphic to the Koszul dual $A^! = T/\mathfrak{r}$, where ideal \mathfrak{r} is generated by relations (5.1).*

It is a standard fact about Koszul duality (see [39]) that if A is commutative, then $A^!$ is a universal enveloping of a graded Lie algebra. In my case, I denote this Lie algebra by L . It is generated by the odd vector space $S_+[\delta, \delta']$. Then

$$A^! = U(L), \quad (5.3)$$

where U stands for the universal enveloping algebra.

Fix a graded vector space $V = \bigoplus_{i \leq 0} V_i$. I use standard notations in homological algebra. $V[1]_i = V_{i+1}$. Denote by

$$\begin{aligned} \text{Sym}(V[1]) &= k \oplus (V_0) \oplus (V_{-1} \oplus V_0 \wedge V_0) \oplus (V_{-2} \oplus V_{-1} \otimes V_0 \oplus V_0 \wedge V_0 \wedge V_0) \oplus \dots \\ &= \text{Sym}(V[1])_0 \oplus \text{Sym}(V[1])_{-1} \oplus \dots \end{aligned}$$

the graded exterior algebra. Chevalley-Eilenberg differential graded algebra is $\text{Hom}(\text{Sym}(L[1]), \mathbb{C})$. Its differential satisfies graded Leibniz rule. On generators $t^a \in \text{Hom}(L, \mathbb{C})$ it is defined as

$$dt^a = -\frac{1}{2}C_{bc}^a t^b \wedge t^c.$$

By construction, the differential in such grading has the degree $+1$. If I follow this practice, the linear space $\text{Hom}(L_1, \mathbb{C}) \cong A_1$ will have cohomological degree two. For me, it will be convenient to make the suspension shift in the opposite direction

$$CE := \text{Hom}(\text{Sym}(L[-1]), \mathbb{C}). \quad (5.4)$$

CE and $\text{Hom}(\text{Sym}(L[1]), \mathbb{C})$ are isomorphic as \mathbb{Z}_2 -graded algebras. The differential in $CE_\bullet = \{CE_\bullet\}_{i \geq 0}$ has degree -1 and $S_+[\delta, \delta']^*_1 \subset P$ acquires degree zero. Components of CE_\bullet are free modules over P .

Proposition 5.5 *The cohomology of CE_\bullet is isomorphic to A .*

Proof. This is a standard fact in the theory of commutative Koszul algebras. In my case, it follows from Proposition 5.2. For details see [39]. ■

5.2 Stanley-Reisner algebra SR

The estimate of the lowest **T**-weight in $H_a^i(A)$ will rely on the following result.

Proposition 5.6 *Fix $[\gamma, \gamma'] \subset [\delta, \delta']$. Let $\mathbf{p}^! : A^![\gamma, \gamma'] \rightarrow A^![\delta, \delta']$ be the map induced by projection $\mathbf{p} : A[\delta, \delta'] \rightarrow A[\gamma, \gamma']$. Then $\mathbf{p}^!$ is injective.*

Proof. The total order (8.2) on \hat{E} induces a total order on $[\delta, \delta'] \subset \hat{E}$. It defines degree-lexicographic filtration $\{F^i\}$ on $A[\delta, \delta']$. It follows directly from straightening law in $A[\delta, \delta']$ that $GrA[\delta, \delta']$ is the Stanley-Reisner algebra $SR[\delta, \delta']$ (see [35] for details). It is the quotient of P by the ideal generated by

$$\lambda^\alpha \lambda^\beta = 0 \quad \forall \alpha, \beta \in [\delta, \delta'] | \alpha \not\prec \beta \text{ and } \alpha \not\succ \beta. \quad (5.7)$$

Suppose that

$$[\gamma, \gamma'] \subset [\delta, \delta']. \quad (5.8)$$

The homomorphism \mathbf{p} induces a homomorphism

$$\mathbf{p} : SR[\delta, \delta'] \rightarrow SR[\gamma, \gamma'].$$

Inclusion (5.8) induces a wrong direction emedding of algebras

$$\mathbf{p}' : SR[\gamma, \gamma'] \rightarrow SR[\delta, \delta']$$

such that $\mathbf{p} \circ \mathbf{p}' = \text{id}$. Thus $SR[\gamma, \gamma']$ is a retract of $SR[\delta, \delta']$.

Denote $A[\delta, \delta']$ by A and $A[\gamma, \gamma']$ by B . We know from [36] that the algebras A and B are Koszul, that is $A^! \cong \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{C}, \mathbb{C})$ and $B^! \cong \bigoplus_{i \geq 0} \text{Ext}_B^i(\mathbb{C}, \mathbb{C})$. I will prove the statement which is dual to injectivity. It is the statement that the map $\mathbf{p}^{!*} : \text{Tor}_i^A(\mathbb{C}, \mathbb{C}) \rightarrow \text{Tor}_i^B(\mathbb{C}, \mathbb{C})$ is onto. Algebras GrA , GrB that defined with respect to filtrations $\{F_A^i\}, \{F_B^i\}$ coincide with $SR[\delta, \delta']$ and $SR[\gamma, \gamma']$. Filtration $\{F_A^i\}$ and $\{F_B^i\}$ defines filtrations on the bar-complexes that compute Tor-groups. These filtrations lead to spectral sequences. By Proposition 7.1 [39], GrA and GrB are also Koszul. Straightening law in the algebra and its Gr implies equality of the Hilbert series $A(t) = GrA(t)$, $B(t) = GrB(t)$. Thus

$$A^!(t) = GrA^!(t) \text{ and } B^!(t) = GrB^!(t). \quad (5.9)$$

The first pages of the spectral sequences are equal to $\text{Tor}^{GrA}(\mathbb{C}, \mathbb{C})$ and $\text{Tor}^{GrB}(\mathbb{C}, \mathbb{C})$ respectively. By Koszulness, the first pages are also equal to $GrA^!$ and $GrB^!$. Equalities of the generating functions (5.9) imply that spectral sequences collapse on the first page.

The map $Gr \mathbf{p}^{!*} : \text{Tor}^{GrA}(\mathbb{C}, \mathbb{C}) \rightarrow \text{Tor}^{GrB}(\mathbb{C}, \mathbb{C})$ has the left inverse $\mathbf{p}^{!*} : \text{Tor}^{GrB}(\mathbb{C}, \mathbb{C}) \rightarrow \text{Tor}^{GrA}(\mathbb{C}, \mathbb{C})$. Thus the map $\mathbf{p}^{!*} : \text{Tor}^A(\mathbb{C}, \mathbb{C}) \rightarrow \text{Tor}^B(\mathbb{C}, \mathbb{C})$ must be surjective. ■

Remark 5.10 *The proof of Proposition 5.6 goes through if I replace the pair of closed intervals by (semi)-open intervals.*

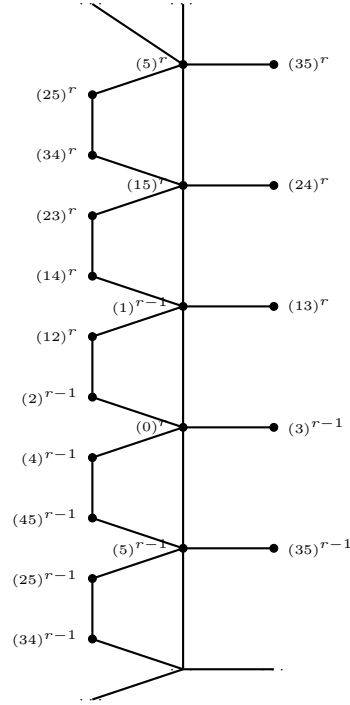
The group Aut acts on $S_+[\delta, \delta']$ and $L[\delta, \delta']$. The action on generators in the notations (2.28) is given by

$$\Pi^*(g)\theta_\alpha := \hat{v}_\alpha^{-1}(t, q, z)\theta_\alpha$$

It is fairly easy to describe Koszul dual $SR^![\delta, \delta']$ by using the general definition of $A^!$ from [39] p.6.

Proposition 5.11 *Algebra $SR^![\delta, \delta']$ has a basis that consists of monomials $\theta_{\alpha_1} \dots \theta_{\alpha_n}$ such that (α_i, α_{i+1}) is a clutter in \hat{E} . The product of two monomials $\theta_{\alpha_1} \dots \theta_{\alpha_n}, \theta_{\beta_1} \dots \theta_{\beta_n}$ is concatenation of monomials if (α_n, β_1) is a clutter and is zero if (α_n, β_1) is not. The algebra $SR^![-\infty, \infty]$ is a path algebra (without local units) of the quiver obtained from the graph (5.12) by replacing each edge by a pair of oppositely oriented edges. $SR^![\delta, \delta']$ is a subalgebra generated by $\theta_\alpha, \alpha \in [\delta, \delta']$*

Proof. Follows directly from (5.7), definition of $A^!$ and Koszul property of $SR^![\delta, \delta']$. I leave derivation of the graph 5.12 from 2.5 to the reader. ■



(5.12)

Any directed path on the diagram 5.12 is characterized by a sequence of vertices $P := (\delta_1, \dots, \delta_n)$ it traverses. The weight $w = (a, u, r)$ of the corresponding monomial $\theta_{\delta_1} \cdots \theta_{\delta_n}$, $n = -a$ in $SR^![\delta, \delta']$ is

$$\left(-n, \sum_{\delta \in P} -u(\delta), \sum_{\delta \in P} -r(\delta) \right).$$

Proposition 5.13 *Dimension of Aut-weight space $SR^![\delta, \delta']^w$, $w = (a, u, r)$ is bounded by a constant C_a that doesn't depend on u and r .*

In addition to that, the number of basis monomials in $SR^![\delta, \delta']$ of degree a that contain $\theta_\alpha \in S_+[\delta, (1)^{-1}]$ and $\theta_\alpha \in S_+[(0)^0, \delta']$ is bounded by the constant C_a .

All the constants do not depend on $[\delta, \delta']$.

Proof. By Proposition 5.11 $SR^![\delta, \delta']^w$ has a basis labelled by paths, whose vertices belong to $[\delta, \delta']$. Let us fix such a path P . The choice of w determines the length $|a|$ (by construction a is negative) of

P. Denote $\underline{u}(P) := \min_P u(\delta), \overline{u}(P) := \max_P u(\delta)$. it follows from the structure of the graph (5.12) that $|u(\delta_i) - u(\delta_{i+1})| \leq 1$. From this I conclude that $\overline{u}(P) - u \leq |a|$, and $u - \underline{u}(P) \leq |a|$. Thus

$$u - |a| \leq \underline{u}(P) \leq \overline{u}(P) \leq u + |a|.$$

There is a finite number $C_{|a|}$ of sequences $P \subset \hat{E}$ that with $|a|$ elements that satisfy these conditions.

To prove the second statement I notice that in this case $\underline{u}(P) \leq 0 \leq \overline{u}(P)$. Thus $-|a| < \underline{u}(P) \leq 0 \leq \overline{u}(P) < |a|$. The rest of the proof repeats the previous arguments. ■

Remark 5.14 *It follows from the proof of Proposition 5.6 that there is an isomorphism of Aut representations $A^![\delta, \delta']$ and $SR^![\delta, \delta']$. There an an isomorphism Aut representations in $L[\delta, \delta']$ and the $LSR^![\delta, \delta']$.*

Proposition 5.15 *Let L^u be \mathbf{T} -weight subspace of $L[\delta, \delta']$. There is a direct sum decomposition $L = L^{\geq 0} + L^{< 0} = \bigoplus_{u \geq 0} L^u + \bigoplus_{u < 0} L^u$. By Proposition 5.6 the Lie subalgebra in $L = L[\delta, \delta']$ that is generated by $S_+[\delta, (1)^{-1}]$ is isomorphic to $L[\delta, (1)^{-1}]$. By construction $L[\delta, (1)^{-1}] \subset L^{> 0}, L[(0)^0, \delta] \subset L^{\leq 0}$. Let $L = \bigoplus_{a \leq 0} L_a$ be \mathbb{C}^\times -grading.*

1. $\dim(L_a^{> 0}/L[\delta, (1)^{-1}]_a) \leq C_a$ $\overline{u}(L_a^{> 0}/L[\delta, (1)^{-1}]_a) \leq C'_a$ for some $C_a, C'_a \geq 0$.
2. Let $w = (a, u, r)$ be an Aut-weight. $\dim_{\mathbb{C}} L^w \leq C''_a$.
3. $\dim_{\mathbb{C}}(L_a^{\leq 0}/L[(0)^0, \delta]_a) \leq D_a$ and $|\overline{u}(L_a^{\leq 0}/L[(0)^0, \delta]_a)| \leq D'_a$ for some $D_a, D'_a \geq 0$

All the constants do not depend on $[\delta, \delta']$.

Proof. Follows from Remark 5.14 and Proposition 5.13. ■

Proposition 5.16

$$\dim_{\mathbb{C}} \text{Ext}_{A[\delta, \delta']}^l(\mathbb{C}, \mathbb{C})^w \leq C_{w, l}$$

where $C_{w, l}$ doesn't depend on δ, δ' .

By Koszul property of $A[\delta, \delta']$ ([36]) $\bigoplus_l \text{Ext}_{A[\delta, \delta']}^l(\mathbb{C}, \mathbb{C}) \cong A^![\delta, \delta']$. The theorem follows from Remark 5.14 and Proposition 5.13.

5.3 Estimates of weights

In this section I will establish a universal bounds on renormalized \mathbf{T} -weights of $H_{\mathfrak{a}}^i[\delta, \delta']$. To do this, I would like interpret the complex CE_{\bullet} (5.4) as a non minimal free resolution of the P -module A . It will be the main technical tool which enable me to find the lower bound on weights of \mathbf{T} -action in $H_{\mathfrak{a}}^i[\delta, \delta']$.

I will start with discussion of dimensions of $\mathbb{C}^{\times} \times \mathbf{T}$ -weight spaces of local cohomology of the free commutative algebra $\mathbb{C}[X]$, $X \subset \hat{E}$. Fix an ideal $\mathfrak{a} = \mathfrak{i}[-\infty, (1)^{-1}] \cap X$. Obviously $H_{\mathfrak{a}}^i(\mathbb{C}[X]) \neq \{0\}$, $i \neq [-\infty, (1)^{-1}] \cap X$. The cohomology can be computed by using K uneth formula and an isomorphism (A.11). I conclude that the character of $\mathbb{C}^{\times} \times \mathbf{T}$ action is equal to

$$\theta_{\mathfrak{a}}^{bare}[X] = \prod_{\alpha^i \in [-\infty, (1)^{-1}] \cap X} \frac{t^{-1}q^{-i}}{(1-t^{-1}q^i)} \prod_{\alpha^i \in [(0)^0, \infty] \cap X} \frac{1}{(1-tq^i)}$$

The renormalized character $\theta_{\mathfrak{a}}[X](t, q) := \theta_{\mathfrak{a}}^{bare}[X](t, q) \prod_{\alpha^i \in [-\infty, (1)^{-1}] \cap X} tq^i$ has a limit when $X = [\delta, \delta']$ and $\delta \rightarrow -\infty, \delta' \rightarrow \infty$:

$$\theta_{\mathfrak{a}}(t, q) = \prod_{i=1}^{\infty} \frac{1}{(1-t^{-1}q^i)^{16}} \prod_{i=0}^{\infty} \frac{1}{(1-tq^i)^{16}} \in \mathbb{Z}((t))[[q]]$$

Fix a notation

$$Chs_{\bullet}^{\mathfrak{c}} := CE_{\bullet}[\delta, \delta'] \otimes_P T_{\mathfrak{c}}[\delta, \delta'], \mathfrak{c} = \mathfrak{a}, \mathfrak{b}, \mathfrak{m}.$$

Proposition 5.17 *Let $b_{\mathfrak{c}} := s(\mathfrak{c}) - s'(\mathfrak{c})$ (see Proposition 4.85 for notations).*

1.

$$H_{\mathfrak{c}}^i(A) = H_{i'}(Chs_{\mathfrak{c}}^{\mathfrak{c}}), i + i' = s(\mathfrak{c})$$

$$H_{b_{\mathfrak{c}}+i}(Chs_{\mathfrak{c}}^{\mathfrak{c}}) = 0, i \neq 0, \dots, 3, \mathfrak{c} = \mathfrak{a}, \mathfrak{b}, i \neq 0 \text{ for } \mathfrak{c} = \mathfrak{f}, \mathfrak{f}', \mathfrak{m}.$$

2. Define

$$r(i) = \begin{cases} 0 & i \leq 0 \\ 1 & i = 1, 2 \\ 2 & i \geq 3 \end{cases}$$

In the notations of Definition 4.63 $\underline{u}\left(H_{\mathfrak{a}}^{3-\rho(\delta)-i}(A)\right) \geq -r(i)$, where I consider the renormalized \mathbf{T} -action (see Remark 4.96). Likewise, $\overline{u}\left(H_{\mathfrak{b}}^{\rho(\delta')+1-i}(A)\right) \leq -r(i)$.

More generally, if only $\delta \in M_1^+ \sqcup M_3^+$ and δ' is arbitrary, then there is i, δ, δ' -independent constant C such that $\underline{u}\left(H_{\mathfrak{a}}^i(A)\right) \geq C$. Similar in case only $\delta' \in M_1^- \sqcup M_3^-$, $\exists C'$ such that $\overline{u}\left(H_{\mathfrak{b}}^{\rho(\delta)-2+i}(A)\right) \leq C'$.

3. Let $H_{\mathbf{c}}^i[\delta, \delta']^w$ be the weight subspace for the renormalized action of Aut . Then

$$\exists C_w \geq 0 \text{ such that } \dim H_{\mathbf{c}}^i[\delta, \beta]^w \leq C_w, \mathbf{c} = \mathbf{a}, \mathbf{b} \text{ for any } [\delta, \beta]. \quad (5.18)$$

Proof.

1. The first statement is a corollary of Proposition 3.30. It is different from Proposition 4.85 only in the choice of resolution.
2. By Proposition 5.6, the map $A[\delta, \delta'] \rightarrow A[\delta, (1)^{-1}]$ induces inclusion of universal enveloping algebras $A^![\delta, (1)^{-1}] \rightarrow A^![\delta, \delta']$ and of the Lie algebras $L[\delta, (1)^{-1}] \subset L[\delta, \delta']$. Let Ker be the kernel of the homomorphism $CE[\delta, \delta'] \rightarrow CE[\delta, (1)^{-1}]$. The powers $\{\text{Ker}^{\times k}\}$ define decreasing filtration of $CE_{\bullet}[\delta, \delta']$ (of Serre-Hochschild type). It defines a filtration in $Chs_{\bullet}^{\mathbf{a}}[\delta, \delta']$. Introduce a notation

$$B_j^k := \text{Hom}_j \left(\text{Sym}^k \left((L[\delta, \delta'] / L[\delta, (1)^{-1}])[-1] \right), \mathbb{C} \right).$$

Gradation j is defined in equation (5.4). $L[\delta, \delta'] / L[\delta, (1)^{-1}]$ and $B^k := \oplus_j B_j^k$ are $L[\delta, (1)^{-1}]$ -module, where the generators are acting by commutations with $\theta_{\alpha}, \alpha \in [\delta, (1)^{-1}]$.

By the first item of this proposition and Theorem 4.11, the only nontrivial cohomology group in $Chs^{\mathbf{a}}[\delta, (1)^{-1}]$ is

$$H_{|[\delta, (1)^{-1}]| + \rho(\delta) - 3}(Chs^{\mathbf{a}}[\delta, (1)^{-1}]) \cong H_{\mathbf{a}}^{3 - \rho(\delta)}[\delta, (1)^{-1}].$$

The initial page of the spectral sequence $E_{sk}^1 = H_{s+k}(Chs^{\mathbf{a}}[\delta, (1)^{-1}] \otimes B^k) \Rightarrow H_{s+k}(Chs^{\mathbf{a}}[\delta, \delta'])$ is equal to cohomology of the complex

$$H_{|[\delta, (1)^{-1}]| + \rho(\delta) - 3}(Chs^{\mathbf{a}}[\delta, (1)^{-1}]) \otimes B_{\bullet}^k.$$

The differential is of Koszul type $\sum_{\alpha \in [\delta, (1)^{-1}]} \lambda^{\alpha} \otimes \theta_{\alpha}$.

By Lemma 4.141 item 3, the lower bound on (nonrenormalized) weights of $H_{\mathbf{a}}^{3 - \rho(\delta)}[\delta, (1)^{-1}] = H_{\mathbf{m}}^{3 - \rho(\delta)}[\delta, (1)^{-1}]$ is $u[\delta, (1)^{-1}]$.

Let

$$K \subset Chs_{|[\delta, (1)^{-1}]| + \rho(\delta) - 3}^{\mathbf{a}}[\delta, (1)^{-1}]$$

be a \mathbf{T} -sub-representation, consisting of cycles, that projects isomorphically $H_{|[\delta, (1)^{-1}]| + \rho(\delta) - 3}(Chs^{\mathbf{a}})$.

The group $H_{\mathbf{a}}^i[\delta, \delta']$ must be isomorphic to a sub-quotient of $K \otimes B^k$.

Denote by TL the Lie subalgebra $\bigoplus_{i \leq -2} L_i \subset L$. Notations $L^<$ is explained in Proposition 5.15. Denote

$$N := TL[\delta, \delta'] / TL[\delta, (1)^{-1}] [-1], \quad M_j := \bigoplus_{t_s | \sum_{s \geq 1} s t_s = -j} \bigotimes \text{Hom}(\text{Sym}^{t_s} N_s, \mathbb{C}).$$

The groups $M_j^{<0}$ is constructed by the same formula with N_s replaced by $N_s^{>0}$ (taking Hom changes signs of weights). Likewise $M_j^{\geq 0}$ is constructed from $N_s^{\leq 0}$. There is an isomorphism

$$B_j \cong \bigoplus_{j_1 + j_2 = j} M_{j_1}^{<0} \otimes M_{j_2}^{\geq 0} \otimes \text{Hom}(\text{Sym} S_+[(0)^0, \delta'], \mathbb{C})$$

By Proposition 5.15 dimensions of $N_s^{>0}$ and of $M_j^{<0}$ are bounded from above by the constants that don't depend on $[\delta, \delta']$. By the same proposition the \mathbf{T} -weights of $N_s^{>0}$ (and automatically the weights of $M_j^{<0}$) are bounded from below by $[\delta, \delta']$ -independent constant C_j . In addition $\underline{u}(M_{j_2}^{\geq 0} \otimes \text{Hom}(\text{Sym} S_+[(0)^0, \delta'], \mathbb{C})) \geq 0$. From this I derive that $C_j \leq \underline{u}(B_j)$. This implies that for the renormalized \mathbf{T} -action $\underline{u}(H_{\mathbf{a}}^{-\rho(\delta)+3-j}[\delta, \delta']) \geq C_j$. The cohomology groups are non-trivial only in the finite range of j . The universal bound is $\min C_j$. Similar arguments work for $\overline{u}(H_{\mathbf{b}}^{\rho(\delta)-2+j}[\delta, \delta'])$.

One can obtain a more accurate bounds on the weight of renormalized \mathbf{T} -action in assumptions that $[\delta, \delta']$ is Gorenstein (2.42). As $\underline{u}(B_0) = \underline{u}(\text{Hom}(\text{Sym} S_+[(0)^0, \delta'], \mathbb{C})) = 0$ the previous arguments imply that $\underline{u}(H_{\mathbf{a}}^{-\rho(\delta)+3}[\delta, \delta']) \geq 0$. $B_1 = (M_1^{<0} + M_1^{\geq 0}) \otimes \text{Hom}(\text{Sym} S_+[(0)^0, \delta'], \mathbb{C})$. Computation with the path algebra from Proposition 5.11 shows that $\dim_{\mathbb{C}} M_1^{<0} = 10$ and all the elements in $M_1^{<0}$ have \mathbf{T} -weight -1 . This verifies the bound for $H_{\mathbf{a}}^{-\rho(\delta)+2}[\delta, \delta']$. Similar arguments establish the upper bounds for $H_{\mathbf{b}}^{\rho(\delta')+1}[\delta, \delta']$ and $H_{\mathbf{b}}^{\rho(\delta')}[\delta, \delta']$.

Finally, I use Poincaré duality between pairs $(H_{\mathbf{b}}^{\rho(\delta')+1}, H_{\mathbf{a}}^{(-\rho(\delta)+3)-3})$ and $(H_{\mathbf{b}}^{\rho(\delta')}, H_{\mathbf{a}}^{(-\rho(\delta)+3)-2})$, Proposition 4.90 and Remark 4.96 to verify the bounds for $\underline{u}(H_{\mathbf{a}}^{(-\rho(\delta)+3)-i})$, $i = 2, 3$.

3. I will prove the statement only for $\mathbf{c} = \mathbf{a}$. By the previous item the group $H_{\mathbf{a}}^{-\rho(\delta)+3}(A)$ is a subquotient of the tensor product $H_{\mathbf{m}}^{3-\rho(\delta)}[\delta, (1)^{-1}] \otimes B_0$. The linear space $H_{\mathbf{m}}^{3-\rho(\delta)}[\delta, (1)^{-1}]$ is dual to $A[\delta, (1)^{-1}]$ (4.12). The later space has a basis formed by standard monomials. It determines a dual weight basis $H_{\mathbf{m}}^{3-\rho(\delta)}[\delta, (1)^{-1}]$. After taking renormalization of Aut action into account I use the dual basis to define an Aut -equivariant embedding of $H_{\mathbf{m}}^{3-\rho(\delta)}[\delta, (1)^{-1}]$ into $\mathbb{C}[\delta, (1)^{-1}]^{-1}$. I conclude that $\chi_{H_{\mathbf{m}}^{3-\rho(\delta)}[\delta, (1)^{-1}]}(t, q) \in \mathbb{Z}[t, t^{-1}][[q]]$. By combining this observation with the results of the previous item I get Aut -equivariant linear embedding of $H_{\mathbf{a}}^{-\rho(\delta)+3}(A)$ into $\mathbb{C}[\delta, (1)^{-1}]^{-1} \otimes \mathbb{C}[(0)^0, \delta']$. Thus $\chi_{H_{\mathbf{a}}^{-\rho(\delta)+3}(A)}(t, q) \in \mathbb{Z}((t))[[q]]$.

Definition 5.19 *I will be using a partial order on series with real coefficients. It is defined by the rule $f(t, q) \leq g(t, q)$ iff $g(t, q) - f(t, q)$ has positive series coefficients. Obviously, if $f(t, q) = \sum_{i,j=-\infty}^{\infty} c_{ij} t^i q^j, c_{ij} \in \mathbb{R}^{\geq 0}, g \in \mathbb{R}((t))[[q]]$ and $f \leq g$ then $f \in \mathbb{R}((t))[[q]]$.*

I conclude $\theta_{\mathbf{a}}(t, q) \geq \chi_{H_{\mathbf{a}}^{-\rho(\delta)+3}(A)}(t, q)$. As $\theta(t, q)$ doesn't depend on $[\delta, \delta']$ this proves the claim for $H_{\mathbf{a}}^{-\rho(\delta)+3}$. The combination of the previous argument with the the proof of the last item leads to inequality.

$$\chi_{H_{\mathbf{a}}^{-\rho(\delta)+3-i}}(t, q) \leq \theta_{\mathbf{a}}(t, q) \sum_{j_i+j_2=i} \chi_{M_{j_1}^{<0}}(t, q) \chi_{M_{j_2}^{\geq 0}}(t, q)$$

$M_{j_2}^{\geq 0}$ is a graded components in a free graded commutative algebra. Though dimension of the generators is infinite, by Proposition 5.15 the generating function is bounded by $\sum_{i=2}^j \frac{C_i t^i}{1-q}$. From this I derive a bound

$$\chi_{M_j^{\geq 0}} \leq \prod_{s=2}^j \prod_{k \geq 0} (1 - (-1)^s t^k q^k)^{(-1)^{s+1} C_s} = \theta_j''(t, q)$$

It follows from from the same proposition that $M_{j_1}^{<0}$ is a graded subspace in a free graded commutative algebra on a finite number of generators. Proposition 5.15 gives a $[\delta, \delta']$ -independent bound on this number. Thus $\chi_{M_j^{<0}} \leq \theta_j'(t, q) \in \mathbb{Z}[t, t^{-1}, q, q^{-1}]$. Finally

$$\chi_{H_{\mathbf{a}}^{-\rho(\delta)+3-i}}(t, q) \leq \theta_{\mathbf{a}}(t, q) \sum_{j_i+j_2=i} \theta_{j_1}'(t, q) \theta_{j_2}''(t, q) \in \mathbb{Z}((t))[[q]]$$

where the right-hand-side doesn't depend on $[\delta, \delta']$.

■

Remark 5.20 Here some interpretation of coefficients of the series (4.122). The -1 in the numerator of the coefficient of q^0 accounts for the generator ω_3 in $H_{\mathbf{a}}^{3-\rho(\delta)}[\delta, \delta']$ with the smallest \mathbf{T} -weight. In my choice of renormalization the weight of ω_3 is set to zero. The group $H_{\mathbf{a}}^{-\rho(\delta)}[\delta, \delta']$ contains a homogeneous element ω_0 that pairs nontrivially with ω_3 . By taking renormalized degree of the pairing into account I conclude that ω_3 has $\mathbb{C}^\times \times \mathbf{T}$ weight equal to $(4, -2)$. I interpret ω_0 as being responsible for the unit in the denominator of the coefficient of q^{-2} .

Note that up to division on t^4 and shift $Z_i \rightarrow Z_{i-2}$ the coefficients that I found coincide with $Z_0(t)$, $Z_1(t)$ and $Z_2(t)$ formulas (3.12), (3.22) and (3.32) [2].

Here is the final result.

Proposition 5.21 *For every Aut-weight $w = (a, u, r)$ the maps*

$$\begin{aligned} \mathbf{p} : H_{\mathbf{b}}^i[\delta, \beta]^w &\rightarrow H_{\mathbf{b}}^i[\delta', \beta]^w, \delta \leq \delta' \leq \beta, \text{ is onto if } u(\delta') < u \\ \mathbf{p} : H_{\mathbf{a}}^i[\delta, \beta']^w &\rightarrow H_{\mathbf{a}}^i[\delta, \beta]^w, \delta \leq \beta \leq \beta', \text{ is onto if } u(\beta) > u \end{aligned} \quad (5.22)$$

Proof. I will prove the statement only for ideal \mathbf{b} . First I assume that $\delta \in M_1^+ \sqcup M_3^+$ and $\delta' \in M_1^+$. Lemma 4.29 allows to decompose \mathbf{p} into a product of elementary projections \mathbf{p}_i . It is sufficient to prove the statement for $\mathbf{p} = \mathbf{p}_i$. Possible \mathbf{p}_i are described in Lemma 4.24. If \mathbf{p} is a regular homomorphism (equations (4.25) and (4.26)) then \mathbf{p} is a map in the short exact sequence

$$\{0\} \rightarrow A[\delta, \beta] \xrightarrow{\lambda^\delta \times ?} A[\delta, \beta] \xrightarrow{\mathbf{p}} A(\delta, \beta) \rightarrow \{0\}.$$

It leads to the long exact sequence of \mathbf{T} -weight components of local cohomology

$$\dots \rightarrow H_{\mathbf{b}}^i[\delta, \beta]^{u-u(\delta)} \xrightarrow{\lambda^\delta \times ?} H_{\mathbf{b}}^i[\delta, \beta]^u \xrightarrow{\mathbf{p}} H_{\mathbf{b}}^i(\delta, \beta)^u \rightarrow H_{\mathbf{b}}^{i+1}[\delta, \beta]^{u-u(\delta)} \rightarrow \dots \quad (5.23)$$

By Proposition 5.17 $\overline{u}(H_{\mathbf{b}}^i[\delta, \beta]) \leq 0$. The (renormalized) \mathbf{T} weight of λ^δ is $u(\delta)$. If the condition (5.22) is satisfied the group $H_{\mathbf{b}}^i[\delta, \beta]^{u-u(\delta)}$ is zero and \mathbf{p} in (5.23) is an isomorphism.

An irregular homomorphism (4.27) \mathbf{p} defines a short exact sequence

$$\{0\} \rightarrow \mathfrak{e}(\delta, \beta) \rightarrow A(\delta, \beta) \xrightarrow{\mathbf{p}} A(\delta', \beta) \rightarrow \{0\}, \quad \rho(\delta') = \rho(\delta'') = \rho(\delta) + 1, \delta'' \in M_2^+, \delta, \delta' \in M_1^+$$

of $\mathbb{C}(\delta, \beta)$ -modules. It is explained in the proof of the second part of Lemma 4.24 that the ideal $I(\delta, \beta)$ is generated by $\lambda^\gamma, \lambda^{\gamma'}, \gamma \leq \gamma' \in \text{CL}^-(\delta')$. There a projection $\mathbf{q} : A(\delta, \beta) \rightarrow A[\gamma, \beta]$. In the same lemma it is also explained that $\mathfrak{e}(\delta, \beta)$ maps isomorphically by \mathbf{q} to ideal $\mathfrak{e}[\gamma, \beta]$ in $A[\gamma, \beta]$. $\mathfrak{e}[\gamma, \beta]$ is generated by $\lambda^\gamma, \lambda^{\gamma'}$. The isomorphism $H_{\mathbf{b}}^i[\gamma, \beta] \cong H_{\mathbf{b}}^i(\mathfrak{e}[\gamma, \beta])$ is established in Proposition E.12 item (4). Thus the composition $H_{\mathbf{b}}^i(\mathfrak{e}(\delta, \beta)) \xrightarrow{j} H_{\mathbf{b}}^i(\delta, \beta) \rightarrow H_{\mathbf{b}}^i[\gamma, \beta]$ is an isomorphism. It implies that j is an inclusion, the boundary differential in the long exact sequence

$$\dots \rightarrow H_{\mathbf{b}}^i(\mathfrak{e}(\delta, \beta)) \rightarrow H_{\mathbf{b}}^i(\delta, \beta) \xrightarrow{H(\mathbf{p})} H_{\mathbf{b}}^i[\delta', \beta] \rightarrow \dots$$

is trivial and $H(\mathbf{p})$ is surjective.

Suppose that now $\delta \in M_1^+ \sqcup M_3^+$ and $\delta' \in M_3^+$. The diagram (2.5) helps to find $\delta'' \in M_1^+$ such that $\delta' < \delta''$. The composition of the maps $H_{\mathbf{b}}^i[\delta, \beta] \xrightarrow{\mathbf{p}} H_{\mathbf{b}}^i[\delta', \beta] \xrightarrow{\mathbf{q}} H_{\mathbf{b}}^i[\delta'', \beta]$ coincides with $H_{\mathbf{b}}^i[\delta, \beta] \xrightarrow{\mathbf{r}} H_{\mathbf{b}}^i[\delta'', \beta]$. Maps \mathbf{p} and \mathbf{r} are surjective. Then so is \mathbf{q} .

■

6 The limiting space of states

Closer to the end this section, I will define the limiting groups $H_a^{i+\frac{\infty}{2}}$ and establish its basic properties. It is also desirable to have a simple model for practical computations with $H_a^i[\delta, \delta']$ and $H_a^{i+\frac{\infty}{2}}$.

The bidirect system of complexes $\text{Fock}_\bullet^a[\delta, \delta']$ whose cohomology coincide with $H_a^i[\delta, \delta']$ serves this purpose. I will devote the next subsection to this complex. Throughout this section all the intervals satisfy purity condition (2.42).

6.1 $\text{Fock}_\bullet^a[\delta, \delta']$ as a function of δ and δ'

In Section 4.2.4, I introduced the complex $\text{Fock}_\bullet^a[\delta, \delta']$ (4.98). My plan is to define a structure bidirect system on $\{\text{Fock}_\bullet^a[\delta, \delta']\}$ whose limit will be Fock_\bullet^a . The important difference from the system $H_a^i[\delta, \delta']$ (4.132), that I already have, is that the bidirect system will now be defined on the level of chains. This explains my interest in functorial properties of the correspondence $[\delta, \delta'] \Rightarrow \text{Fock}_\bullet^c[\delta, \delta']$.

Recall that there is identification of cohomology of $\text{Fock}_\bullet^c[\delta, \delta']$ and $H_a^i[\delta, \delta']$ (4.100). One can think about the correspondence

$$[\delta, \delta'] \Rightarrow H_a^i[\delta, \delta'] \quad (6.1)$$

as a functor from the category of intervals $\{[\delta, \delta']\} (\delta' \text{ is fixed})$ to linear spaces. th (3.5) are the structure maps of this functor. The question I will address now is how to see the lift Th_p of th_p to the level of chains of $\text{Fock}_\bullet^a[\delta, \delta']$.

I am going to use one more time the familiar idea that th can be decomposed into product of elementary th_{p_i} . This time I will lift th_{p_i} on the level of chains. Presently I will define such a lifting, which I denote by Th_{p_i} . For this purpose I will applied Lemma 4.29 to the pair $[\delta, \beta] \supset [\delta', \beta]$ to construct maps p_i (4.31). From this lemma I know that there are three kinds of homomorphisms p_i (4.25, 4.26, 4.27). I will deal with the regular cases (4.25, 4.26) first.

The regular case Let us assume that δ is the same as one of the left-end points of the intervals (4.25, 4.26). This means that λ^δ is not a zero divisor in $A[\delta, \delta']$ and $A[\delta, \delta']/(\lambda^\delta) \cong A(\delta, \delta')$. I define the map $\text{Th} : I_a(\delta, \delta') \rightarrow I_a[\delta, \delta']$ on the decomposable tensors $a \otimes_{R^a} f \in A \otimes_{R^a} S_a$ by the formula

$$\text{Th}(a \otimes_{R^a} f) := \tilde{a} \otimes_{R^a} k(f) = \tilde{a} \otimes_{R^a} \frac{1}{\lambda^\delta} f \quad (6.2)$$

where the element \tilde{a} is any preimage a in $A[\delta, \delta']$ with respect to p (4.127), k is taken from Definition 4.72. $\frac{1}{\lambda^\delta} = \varpi_{\text{Im}(p_{\text{roinc}})^\perp}$ in the formula (C.3), that describes the map k . As $\lambda^\delta \frac{1}{\lambda^\delta} f = 0 \in S_a[\delta, \delta']$, it follows

that Th doesn't depend on the possible choices of \tilde{a} . I extend this to a map of free $\Lambda[\theta^0, \theta^1, \theta^2]$ -modules $\text{Fock}_i^a(\delta, \delta') \rightarrow \text{Fock}_i^a[\delta, \delta']$.

The irregular case Let us assume that $\delta \in M_2^-$ (Definition E.2), which means that δ is one of the left-end points of the closed intervals (4.27). γ is the corresponding left-end point of the open interval.

The map $\text{Th} : I_a[\delta, \delta'] \rightarrow I_a(\gamma, \delta')$ on a decomposable element $a \otimes_P f$ is defined as

$$\text{Th}(a \otimes f) := v(a) \otimes_{R^a} i(f) = \tilde{a} \otimes_{R^a} k(f) \quad (6.3)$$

where v and k are introduced in equation (4.129) and Definition 4.72 respectively. As in the regular case, I extend this to a map of free $\Lambda[\theta^0, \theta^1, \theta^2]$ -modules $\text{Fock}_i^a(\gamma, \delta') \rightarrow \text{Fock}_i^a[\delta, \delta']$.

Th in the general case

Definition 6.4 I assume that $[\delta, \beta] \supset [\delta', \beta]$.

1. I define the map

$$\text{Th} : \text{Fock}_i^a[\delta', \beta] \rightarrow \text{Fock}_i^a[\delta, \beta] \quad (6.5)$$

as a composition

$$\text{Fock}_i^a[\Delta_{n'}] \xrightarrow{\text{Th}_{p_{n'}-1}} \dots \xrightarrow{\text{Th}_{p_1}} \text{Fock}_i^a[\Delta_1].$$

where p_i are taken from (4.31). To construct (semi)intervals Δ_i I applied Lemma 4.29 to $[\delta, \beta] \supset [\delta', \beta]$.

2. Suppose $\beta' < \beta$. The map

$$\underline{p} : \text{Fock}_i^a[\delta, \beta] \rightarrow \text{Fock}_i^a[\delta, \beta'] \quad (6.6)$$

is induced by the maps of the tensor components of the module I_a

$$p : A[\delta, \beta] \rightarrow A[\delta, \beta'] \text{ and } pr : S_a[\delta, \beta] \rightarrow S_a[\delta, \beta'] (4.76).$$

The map (6.6) is also defined for semi-intervals $(\delta, \beta] \supset (\delta, \beta']$.

Remark 6.7 The maps Th and \underline{p} commute.

Compatibility of Th and th is verified in the following statement.

Proposition 6.8 In the assumptions of Proposition 4.124 ($\beta = \delta_3, \delta_4$) and after identification (4.100), (4.106)

1. The map $\text{Th}_{\mathbf{q}}$ on $I_{\mathbf{a}}$ coincides with

$$\text{th}_{\mathbf{q}} \otimes \text{id} : H_{\mathbf{f}}^{-\rho(\delta_2)}[\delta_2, \beta] \otimes \mathbb{C}[A^{\mathbf{a}}]^{-1} \rightarrow H_{\mathbf{f}}^{-\rho(\delta_1)}[\delta_1, \beta] \otimes \mathbb{C}[A^{\mathbf{a}}]^{-1}$$

2. The induced map

$$\text{Th}_{\mathbf{q}} : H_j(\text{Fock}^{\mathbf{a}}[\delta_2, \beta]) \rightarrow H_j(\text{Fock}^{\mathbf{a}}[\delta_1, \beta]), \delta_1 < \delta_2$$

coincides with the map

$$\text{th}_{\mathbf{q}} : H_{\mathbf{a}}^{3-\rho(\delta_2)-j}[\delta_2, \beta] \rightarrow H_{\mathbf{a}}^{3-\rho(\delta_1)+j}[\delta_1, \beta].$$

3. The map

$$\underline{\mathbf{p}} : H_j(\text{Fock}^{\mathbf{a}}[\delta, \beta']) \rightarrow H_j(\text{Fock}^{\mathbf{a}}[\delta, \beta]), \beta < \beta' (\beta = \delta_3, \beta' = \delta_4)$$

coincides with the map

$$\mathbf{p} : H_{\mathbf{a}}^{3-\rho(\delta)-j}[\delta, \beta'] \rightarrow H_{\mathbf{a}}^{3-\rho(\delta)+j}[\delta, \beta].$$

Proof. I start with the proof of the second statement. For this purpose, I replace $[\delta', \beta] \supset [\delta, \beta]$ by elementary extensions of sets $\Delta_i \supset \Delta_{i+1}$ from Definition 6.4. If $\mathbf{q} : A(\Delta_i) \supset A(\Delta_{i+1})$ belongs to the groups (4.25), (4.26), then, by Proposition 3.9, $\text{th}_{\mathbf{q}}$ is the class of the extension (4.127).

By (3.18), th is the boundary homomorphism $H_{\mathbf{a}}^i[\delta, \beta] \rightarrow H_{\mathbf{a}}^{i+1}(\delta, \beta]$ corresponding to (4.127). I replace $A[\delta, \beta]$ in (4.127) by its $\mathbb{C}[\delta, \beta]$ resolution $F_{\bullet}[\delta, \beta]$. The extension lifts to an exact triangle

$$F_{\bullet} \xrightarrow{\lambda^{\delta} \times ?} F_{\bullet} \rightarrow B_{\bullet}(F, \lambda^{\delta}) \rightarrow F_{\bullet}[-1].$$

$B_{\bullet}(F, \lambda^{\delta})$ is the diagonal complex of the Koszul (bi)-complex $B_{\bullet}(F_{\bullet}, \lambda^{\delta})$ which coincides with the cone of the operator $\lambda^{\delta} \times ?$ on F_{\bullet} . The diagonal complex $\text{Fock}_{\bullet}^{\mathbf{a}}(F)$ of the bicomplex $\text{Fock}_{\bullet}^{\mathbf{a}}(F_{\bullet})$ is a modification of $\text{Fock}_{\bullet}^{\mathbf{a}}$ in which A in (4.97) is replaced F_{\bullet} . Evidently, $\text{Fock}_{\bullet}^{\mathbf{a}}$ and $\text{Fock}_{\bullet}^{\mathbf{a}}(F)$ are quasi-isomorphic. My task is to trace the action of the boundary map

$$\text{Fock}_{\bullet}^{\mathbf{a}}(B(F)) = B_{\bullet}(\text{Fock}^{\mathbf{a}}(F)) \rightarrow \text{Fock}_{\bullet}^{\mathbf{a}}(F[-1]).$$

The Koszul construction $B_{\bullet}(\text{Fock}^{\mathbf{a}}(F)) = \text{Fock}^{\mathbf{a}}(F) \otimes \Lambda[\theta^{\delta}]$ is built on the operator $\lambda^{\delta} \times ?$ on $I_{\mathbf{a}}(F_i) = F_i \otimes_R S_{\mathbf{a}}$, which is then extended to $\text{Fock}_{\bullet}^{\mathbf{a}}(F)$. Originally, it is induced from the action of $\lambda^{\delta} \times ?$ on F_i . Notice that $\lambda^{\delta} \times ?$ acting on $S_{\mathbf{a}}$ defines the same operator in $I_{\mathbf{a}}(F_i)$. In the spectral sequence of the bicomplex $B_i((\text{Fock}^{\mathbf{a}} F)_j)$, the zero cohomology $H_0 B((\text{Fock}^{\mathbf{a}} F)_{\bullet})$ is zero, for $\lambda^{\delta} \times ?$ is surjective in $S_{\mathbf{a}}[\delta, \beta]$. The kernel of $\lambda^{\delta} \times ?$ in $\text{Fock}_{\bullet}^{\mathbf{a}} F[\delta, \beta]$ consists of elements of the form $\frac{1}{\lambda^{\delta}} b, b \in \text{Fock}_{\bullet}^{\mathbf{a}} F(\delta, \beta]$. The assignment $b \rightarrow \frac{1}{\lambda^{\delta}} b$ allows me to identify the cohomology of $B_{\bullet}(\text{Fock}^{\mathbf{a}}(F[\delta, \beta]))$ with the cohomology

of $\text{Fock}_\bullet^a(F(\delta, \beta))$. The boundary map takes $\frac{1}{\lambda^s} b \xi \in B_\bullet(\text{Fock}^a(F))$ to $\frac{1}{\lambda^s} b \in \text{Fock}_\bullet^a(F)$. Now it becomes evident that the boundary map will match with (6.2) after I compress resolutions $F_\bullet[\delta, \beta]$ and $F_\bullet(\delta, \beta)$ back to $A[\delta, \beta]$ and $A(\delta, \beta)$.

It remains to understand the structure of the map in local cohomology induced by inclusion $[\delta', \beta] \subset (\delta, \beta]$ (4.129). I study it by replacing local cohomology by more manageable Tor groups. I factor the map into a composition of two maps. The first is

$$H_a^{3-\rho(\delta')-i}[\delta', \beta] = \text{Tor}_i^{\mathbb{C}[\delta', \beta]}(A[\delta', \beta], S_a[\delta', \beta]) \xrightarrow{i} \text{Tor}_i^{\mathbb{C}(\delta, \beta]}(A[\delta', \beta], S_a(\delta, \beta)).$$

The isomorphism is produced by maps $\mathbb{C}[\delta', \beta] \rightarrow \mathbb{C}(\delta, \beta]$, $S_a[\delta', \beta] \rightarrow S_a(\delta, \beta]$ which are induced by i (4.10).

Arguing as in the proof of Proposition 4.100 and taking into account that $[\delta', \beta] \subset (\delta, \beta]$ belong to the class (4.27), I verify that the isomorphism m is induced by the map of the close relative of the complex Fock_\bullet^a :

$$i : B_\bullet(S_a[\delta', \beta] \otimes_{R[\delta', \beta]} A[\delta', \beta], \{\underline{\lambda}^0, \underline{\lambda}^1, \underline{\lambda}^2\}) \rightarrow B_\bullet(S_a(\delta, \beta] \otimes_{R(\delta, \beta]} A[\delta', \beta], \{\underline{\lambda}^0, \underline{\lambda}^1, \underline{\lambda}^2\}). \quad (6.9)$$

The second map in the factorisation

$$\text{Tor}_i^{\mathbb{C}(\delta, \beta]}(A[\delta', \beta], S_a(\delta, \beta]) \xrightarrow{v} \text{Tor}_i^{\mathbb{C}(\delta, \beta]}(A(\delta, \beta], S_a(\delta, \beta]) \cong H_a^{3-\rho(\delta')-i}(\delta, \beta]$$

is induced by the map of $\mathbb{C}(\delta, \beta]$ -modules $\lambda^{\delta'} \times ?$ (4.129). Again, repeating the arguments of the proof of Proposition 4.100, we see that v lifted to the level of map of chains complexes

$$B_\bullet(S_a(\delta, \beta] \otimes_{R(\delta, \beta]} A[\delta', \beta], \{\underline{\lambda}^0, \underline{\lambda}^1, \underline{\lambda}^2\}) \xrightarrow{v} B_\bullet(S_a(\delta, \beta] \otimes_{R(\delta, \beta]} A(\delta, \beta], \{\underline{\lambda}^0, \underline{\lambda}^1, \underline{\lambda}^2\})$$

is equal to $\lambda^{\delta'} \times ?$.

We see that $v \circ i$ coincides with right hand side of (6.3). The first statement uses isomorphism (4.107) and follows the same lines as the second.

I leave the proof of the third statement as an exercise.

■

Example 6.10 *To make an illustration I suppose $k = \rho(\lambda^{\delta_2})$, $k-1 = \rho(\lambda^{\delta'_2})$. Then the map (6.9) does the following transformation*

$$a \otimes \varpi[\delta_2, \beta] = a \otimes \frac{1}{\lambda^{\delta_2}} \frac{1}{\lambda^{\delta'_2}} \prod_{k-1 < i \leq -1} \frac{1}{\lambda^i} \xrightarrow{i} a \otimes \frac{1}{\lambda^k} \frac{1}{\lambda^{k-1}} \prod_{k-1 < i \leq -1} \frac{1}{\lambda^i} = a \otimes \varpi(\delta_1, \beta).$$

Elements λ^i are defined in (4.9). The values of the composition of i and $\lambda^{\delta_2} \times ?$ is

$$a \otimes \frac{1}{\lambda^{\delta_2}} \frac{1}{\lambda^{\delta_2'}} \prod_{k-1 < i \leq -1} \frac{1}{\lambda^i} \rightarrow \tilde{a} \otimes \frac{\lambda^{\delta_2}}{\lambda^k} \frac{1}{\lambda^{k+1}} \prod_{k+1 < i \leq -1} \frac{1}{\lambda^i}.$$

Lemma 6.11 *The maps (6.2) and (6.3) and their compositions (6.5) are injective. The map 6.6 is surjective.*

Proof. Let us consider the regular case (6.2) first. By Propositions 6.8 and 4.105 the map (6.2) is a map of free $\Lambda[\theta^0, \theta^1, \theta^2]$ -modules induced on the generating spaces by the boundary differential in the exact sequence

$$\dots \rightarrow H_{\mathfrak{f}}^{-\rho(\delta)-1}[\delta, \delta'] \rightarrow H_{\mathfrak{f}}^{-\rho(\delta)-1}(\delta, \delta') \rightarrow H_{\mathfrak{f}}^{-\rho(\delta)}[\delta, \delta'] \rightarrow \dots$$

By Corollary 4.108 $H_{\mathfrak{f}}^i[\delta, \delta']$ is nonzero only in degree $-\rho(\delta)$ and $H_{\mathfrak{f}}^i(\delta, \delta')$ is nonzero only in degree $-\rho(\delta) - 1$. Thus the boundary differential must be an inclusion.

Consider now the irregular case. Arguing as in the regular case we see that the map (6.3) is induced by the map

$$H_{\mathfrak{c}}^{-\rho(\delta)-1}[\delta', \beta] \rightarrow H_{\mathfrak{c}}^{-\rho(\delta)-1}(\delta, \beta)$$

It originates from the inclusion map in the short exact sequence $0 \rightarrow A[\delta', \beta] \xrightarrow{\lambda^{\delta_2} \times ?} A(\delta, \beta) \rightarrow A[\delta'', \beta] \rightarrow 0$. Note that $\rho(\delta') = \rho(\delta'')$. We see that the argument of the regular case can be repeated. The long exact sequence of local cohomology is in fact a short exact sequence nontrivial in degree $-\rho(\delta')$. Thus the map in question is an inclusion.

Similar arguments explain surjectivity of the map \underline{p} . The only difference is that the long exact sequences reduce to short exact sequences of local cohomology in degree $-\rho(\delta)$. ■

6.2 The complex $\text{Fock}_{\bullet}^{\mathfrak{a}}$

My present goal is to construct a complex that would compute cohomology $H_{\mathfrak{a}}^{3-i+\frac{\infty}{2}}$ in one step. This complex will be denoted by $\text{Fock}_{\bullet}^{\mathfrak{a}}$.

The maps (6.5) and (6.6) define a bidirect system on $\text{Fock}_{\bullet}^{\mathfrak{a}}[\delta, \gamma]$ parametrized by $\delta \in M_1^+, \gamma \in M_1^-$. I use reductive action of group Aut to decompose this complex into a direct sum

$$\text{Fock}_{\bullet}^{\mathfrak{c}}[\delta, \gamma] = \bigoplus_w \text{Fock}_{\bullet}^{\mathfrak{c}}[\delta, \gamma]^w, \mathfrak{c} = \mathfrak{a}, \mathfrak{b}$$

Here w stands for the weight of Aut .

Introduce notations for limits:

$$\begin{aligned}\overline{\text{Fock}}_{\bullet}^{\mathfrak{c},w} &:= \lim_{\delta \rightarrow} \lim_{\gamma \leftarrow} \text{Fock}_{\bullet}^{\mathfrak{c}}[\delta, \gamma]^w, \\ \underline{\text{Fock}}_{\bullet}^{\mathfrak{c},w} &:= \lim_{\gamma \leftarrow} \lim_{\delta \rightarrow} \text{Fock}_{\bullet}^{\mathfrak{c}}[\delta, \gamma]^w.\end{aligned}\tag{6.12}$$

Proposition 6.13 *The map*

$$\kappa : \overline{\text{Fock}}_{\bullet}^{\mathfrak{c},w} \rightarrow \underline{\text{Fock}}_{\bullet}^{\mathfrak{c},w}, \mathfrak{c} = \mathfrak{a}, \mathfrak{b}\tag{6.14}$$

is an embedding.

Proof. By Lemma 6.11 conditions of Proposition D.16 on the structure maps of the bidirect system are satisfied. ■

Definition 6.15 *Introduce notations for limits:*

$$\begin{aligned}\overline{\text{Fock}}_{\bullet}^{\mathfrak{c},w} &:= \lim_{\delta \rightarrow} \lim_{\gamma \leftarrow} \text{Fock}_{\bullet}^{\mathfrak{c}}[\delta, \gamma]^w, \\ \underline{\text{Fock}}_{\bullet}^{\mathfrak{c},w} &:= \lim_{\gamma \leftarrow} \lim_{\delta \rightarrow} \text{Fock}_{\bullet}^{\mathfrak{c}}[\delta, \gamma]^w.\end{aligned}\tag{6.16}$$

6.3 The definition of semi-infinite local cohomology

Our definition of semi-infinite local cohomology uses bidirect systems of linear spaces. The reader can consult [19],[24], [16], for details on this topic.

Let w be a weight of the group Aut . By Remark 4.133 maps in (4.132) commute with the renormalized Aut action. This is why weight spaces $H_{\mathfrak{c}}^i[\delta, \delta']^w$ form a bidirect system labelled by $\delta \in \mathbf{A}^+ := \mathbf{M}_1^+ \sqcup \mathbf{M}_3^+$ and $\delta' \in \mathbf{A}^- := \mathbf{M}_1^- \sqcup \mathbf{M}_3^-$. The sets of indices have a partial order induced from $\hat{\mathbf{E}}$.

There are several types of semi-infinite local cohomology groups.

Definition 6.17 The Aut weight subspace of semi-infinite local cohomology ($\mathfrak{c} = \mathfrak{a}, \mathfrak{b}$) of the directed system of algebras $A[\delta, \delta']$ (Definition 2.7) corresponding to the weight w are the limits

$$\overline{H}_{\mathfrak{c}}^{i+\frac{\infty}{2},w} := \begin{cases} \lim_{\delta \in \mathbf{A}^+} \lim_{\delta' \in \mathbf{A}^-} H_{\mathfrak{a}}^{i+s'(\mathfrak{a})}[\delta, \delta']^w, \mathfrak{c} = \mathfrak{a} \\ \lim_{\delta \in \mathbf{A}^-} \lim_{\delta' \in \mathbf{A}^+} H_{\mathfrak{b}}^{i+s'(\mathfrak{b})}[\delta, \delta']^w, \mathfrak{c} = \mathfrak{b} \end{cases}, \quad \underline{H}_{\mathfrak{c}}^{i+\frac{\infty}{2},w} := \begin{cases} \lim_{\delta' \in \mathbf{A}^-} \lim_{\delta \in \mathbf{A}^+} H_{\mathfrak{a}}^{i+s'(\mathfrak{a})}[\delta, \delta']^w, \mathfrak{c} = \mathfrak{a} \\ \lim_{\delta' \in \mathbf{A}^+} \lim_{\delta \in \mathbf{A}^-} H_{\mathfrak{b}}^{i+s'(\mathfrak{b})}[\delta, \delta']^w, \mathfrak{c} = \mathfrak{b}. \end{cases}\tag{6.18}$$

The structure maps of the bidirect system $\delta \in M_1^+ \sqcup M_3^+, \delta' \in M_1^- \sqcup M_3^-$ come from the diagram (4.132). The function s' is taken from (4.87). Similar definition can be made using subsets of indices $B^+ := \{(0)^u | u \in \mathbb{Z}\} \subset M_3^+, B^- := \{(1)^u | u \in \mathbb{Z}\} \subset M_3^-$.

There is a canonical homomorphism

$$\kappa_{\mathfrak{c}}^w : \overline{H}_{\mathfrak{c}}^{i+\frac{\infty}{2},w} \rightarrow \underline{H}_{\mathfrak{c}}^{i+\frac{\infty}{2},w}, \mathfrak{c} = \mathfrak{a}, \mathfrak{b}, \quad (6.19)$$

(see (D.10) definition). For more substantial discussion of κ see [19] Section 3.2. Direct and inverse limits usually do not commute. See Example D.12 for discussion of this phenomenon. This is why I can't take for granted that $\kappa_{\mathfrak{c}}$ is an isomorphism. Still it is true:

Proposition 6.20 1. *The map (6.19) is an isomorphism.*

2. *The limits taken over the sets A^+, A^- and B^+, B^- coincide.*

3. *By virtue of the previous item the following notations make sense*

$$\begin{aligned} H_{\mathfrak{c}}^{i+\frac{\infty}{2},w} &:= \overline{H}_{\mathfrak{c}}^{i+\frac{\infty}{2},w} = \underline{H}_{\mathfrak{c}}^{i+\frac{\infty}{2},w} \\ H_{\mathfrak{c}}^{i+\frac{\infty}{2}} &:= \bigoplus_w H_{\mathfrak{c}}^{i+\frac{\infty}{2},w}, \mathfrak{c} = \mathfrak{a}, \mathfrak{b} \end{aligned}$$

The space $H_{\mathfrak{c}}^{i+\frac{\infty}{2}}$ is equipped with the $\text{Spin}(10)$ -action.

Proof. Fix the weight $w = (a, u, r)$. By Proposition 5.21 and *-duality (Proposition 4.138) the map

$$\mathfrak{p} : H_{\mathfrak{a}}^i[\delta, \beta']^w \rightarrow H_{\mathfrak{a}}^i[\delta, \beta]^w, u(\beta) > u \quad (6.21)$$

is onto and the map

$$\text{th}_{\mathfrak{p}} : H_{\mathfrak{a}}^i[\delta, \beta']^w \rightarrow H_{\mathfrak{a}}^{i+\text{codim}}[\delta', \beta']^w, \text{codim} = \rho(\delta) - \rho(\delta'), u < -u(\delta) \quad (6.22)$$

is an injection.

Let $k(\beta')$ be the maximal integer such that for any δ with $\rho(\delta) \leq k(\beta')$ the map (6.22) is an isomorphism. Such k exists because of the bound (5.18). I claim that $\inf_{\beta' > (0)^0} k(\beta') > -\infty$. Let us assume otherwise. Then there is a sequence

$$\beta_1 < \beta_2 < \dots \text{ such that } k(\beta_i) > k(\beta_{i+1}). \quad (6.23)$$

I can assume that $u(\beta_i) > u$ for all i . I pick a sequence δ_i such that $\rho(\delta_i) = k(\beta_i)$. By the assumption (6.23) I have a sequence of maps

$$H_{\mathfrak{a}}[\delta_1, \beta_1]^w \leftarrow H_{\mathfrak{a}}[\delta_1, \beta_2]^w \rightarrow H_{\mathfrak{a}}[\delta_2, \beta_2]^w \leftarrow H_{\mathfrak{a}}[\delta_2, \beta_3]^w \rightarrow \dots$$

In the above formula I omitted cohomological degrees for simplicity. As $k(\beta_{i+1}) < \rho(\delta_i)$ in the diagram inclusions, defined by th , are *never* isomorphisms. This diagram contradicts to the bound (5.18).

I conclude that there is a constant $k = \inf_{\beta' > (0)^0} k(\beta')$ such that the maps (6.22) are isomorphisms if

$$u(\delta) < -u \text{ and } \rho(\delta) < k. \quad (6.24)$$

In these range of indices inverse systems $H_{\mathbf{a}}^i[\delta_i, \beta]^w$ ($i = 1, 2$) are isomorphic. The structure maps of the systems are surjections. By using the bound (5.18) one more time I conclude that there is an integer l such that if

$$\rho(\beta) > l, u(\beta) > u \quad (6.25)$$

and (6.24) hold then (6.21) and (6.22) are isomorphisms. It is clear that the limits of the bidirect system $H_{\mathbf{c}}^{i+s'(\mathbf{a})}[\delta, \beta]^w$ won't change if I restrict the range of indices that satisfy (6.24, 6.25). Such bidirect system satisfies conditions of the Theorem 5.6 [24], from which the first statement follows.

I just constructed a range of indices \mathbf{A}'^{\pm} such that all the structure maps for the bidirect system $H_{\mathbf{a}}^i[\delta, \beta]^w$ are isomorphisms. The double limit is isomorphic to $H_{\mathbf{a}}^i[\delta_0, \beta_0]^w$ $\delta_0 \in \mathbf{A}'^+ \cap \mathbf{M}_3^+$, $\beta_0 \in \mathbf{A}'^- \cap \mathbf{M}_3^0$. Later space is isomorphic to the double limit along $\mathbf{B}'^{\pm} := \mathbf{A}'^{\pm} \cap \mathbf{B}^{\pm}$. Limits over the sets \mathbf{B}'^{\pm} and \mathbf{B}^{\pm} obviously coincide. This proves the second statement of the theorem.

By Remark 4.2 the group acts on $H_{\mathbf{a}}^i[(0)^N, (1)^{N'}]$. It is evident that all the structure maps of the system labelled by \mathbf{B}^{\pm} are compatible with $\text{Spin}(10)$ -action. This proves the last assertion.

■

Definition 6.26 *By virtue of Proposition 6.20 the following notations make sense*

$$\begin{aligned} H_{\mathbf{c}}^{i+\frac{\infty}{2}, w} &:= \overline{H}_{\mathbf{c}}^{i+\frac{\infty}{2}, w} = \underline{H}_{\mathbf{c}}^{i+\frac{\infty}{2}, w} \\ H_{\mathbf{c}}^{i+\frac{\infty}{2}} &:= \bigoplus_w H_{\mathbf{c}}^{i+\frac{\infty}{2}, w}, \mathbf{c} = \mathbf{a}, \mathbf{b} \end{aligned}$$

Corollary 6.27 *Let $w = (a, u, r)$ be an Aut-weight.*

1. $H_{\mathbf{a}}^{i+\frac{\infty}{2}, w} = \{0\}$ for $u < 0$, $H_{\mathbf{b}}^{i+\frac{\infty}{2}, w} = \{0\}$ for $u > 0$. (Proposition 5.17 item 2)
2. $\dim_{\mathbb{C}} H_{\mathbf{c}}^{i+\frac{\infty}{2}, w} < \infty$.
3. Let $Z_{\mathbf{c}}[\delta, \delta']$, $Z_{\mathbf{c}}(t, q, z)$ be the renormalized virtual character (4.110) of $H_{\mathbf{c}}^i[\delta, \delta']$ and $H_{\mathbf{c}}^{i+\frac{\infty}{2}}$ respectively. Then

$$\lim_{\delta \rightarrow -\infty, \delta' \rightarrow \infty} Z_{\mathbf{c}}[\delta, \delta'](t, q, z) = Z_{\mathbf{c}}(t, q, z), \mathbf{c} = \mathbf{a}, \mathbf{b}$$

The limit is understood in the sense of coefficient-wise convergence in $\mathbb{Z}[\widetilde{\mathbf{T}}]((t))((q))$. $\mathbb{Z}[\widetilde{\mathbf{T}}]$ is a group algebra of the group of $\widetilde{\mathbf{T}}$ -characters.

Remark 6.28 *It would be interesting to explore point-wise convergence of*

$$Z_{\mathbf{a}}(t, q, z) = \lim_{N \rightarrow -\infty, N \rightarrow \infty} Z_{\mathbf{a}N}^{N'}(t, q, z)$$

In particular It is true that $Z_{\mathbf{a}}(t, q, z)$ is a meromorphic function on Aut with poles only at $\hat{v}_{\alpha}(t, q, z) = 1$?

Proposition 6.29

$$H_i \underline{\text{Fock}}^{\mathbf{c}} \cong H_i \overline{\text{Fock}}^{\mathbf{c}} \cong H_{\mathbf{c}}^{3-i+\frac{\infty}{2}}, \mathbf{c} = \mathbf{a}, \mathbf{b}$$

Proof. I will give a proof for $\mathbf{c} = \mathbf{a}$. By Proposition 4.99 $H_i(\text{Fock}^{\mathbf{a}}[\delta, \gamma]^w) \cong H_{\mathbf{a}}^{s'(\mathbf{a})-i}[\delta, \gamma]^w$. By Lemma 6.11 $\text{Fock}^{\mathbf{a}}[\delta, \gamma]^w$ with fixed δ satisfies Mittag-Leffler condition. Denote temporarily

$$F_{\gamma, \bullet} := \varinjlim \text{Fock}^{\mathbf{a}}[\delta, \gamma]^w, \quad G_{\gamma}^i := \varinjlim H_{\mathbf{a}}^{s'(\mathbf{a})-i}[\delta, \gamma]^w.$$

The Lemma 6.11 also implies Mittag-Leffler condition for $F_{\gamma, \bullet}$.

Milnor's exact sequence (see e.g. [49]) gives

$$\{0\} \rightarrow \varprojlim^1 H_{\mathbf{a}}^{s'(\mathbf{a})-i-1}[\delta, \gamma]^w \rightarrow H_i(\varprojlim \text{Fock}^{\mathbf{a}}[\delta, \gamma]^w) \xrightarrow{\mathbf{q}} \varprojlim H_{\mathbf{a}}^{s'(\mathbf{a})-i}[\delta, \gamma]^w \rightarrow \{0\}$$

By Proposition 5.21 $H_{\mathbf{a}}^{s'(\mathbf{a})-i}[\delta, \gamma]^w$ satisfies Mittag-Leffler condition and $\varprojlim^1 = \{0\}$. This implies that \mathbf{q} is an isomorphism. Direct limits in the category of linear spaces commute with homology ([49] Theorem 2.6.15). From this I deduce isomorphisms

$$H_i(\overline{\text{Fock}}^{\mathbf{c}, w}) = H_i(\varinjlim \varprojlim \text{Fock}^{\mathbf{a}}[\delta, \gamma]^w) \cong \varinjlim \varprojlim H_{\mathbf{a}}^{s'(\mathbf{a})-i}[\delta, \gamma]^w = \overline{H}_{\mathbf{a}}^{3-i+\frac{\infty}{2}, w} \quad \text{and} \quad (6.30)$$

$$H_i(F_{\gamma}) \cong G_{\gamma}^i.$$

Again by Proposition 5.17 $\dim_{\mathbb{C}} G_{\gamma}^i < C_w$. This automatically implies Mittag-Leffler condition for G_{γ}^i . By using Milnor sequence one more time I get an isomorphism

$$H_i(\underline{\text{Fock}}^{\mathbf{c}, w}) = H_i(\varinjlim F_{\gamma}) \cong \varinjlim G_{\gamma}^i = \underline{H}_{\mathbf{a}}^{3-i+\frac{\infty}{2}, w} \quad (6.31)$$

The proof follows from isomorphisms (6.14), (6.30), and (6.31). ■

From now on

$$H_{\mathbf{a}}^{i+\frac{\infty}{2}} := \overline{H}_{\mathbf{a}}^{i+\frac{\infty}{2}} \cong \underline{H}_{\mathbf{a}}^{i+\frac{\infty}{2}}$$

The pairing on $H_{\mathbf{a}}^{i+\frac{\infty}{2}}$ will be introduced presently.

Proposition 6.32 *There is a nondegenerate pairing*

$$H_{\mathbf{a}}^{i+\frac{\infty}{2}} \otimes H_{\mathbf{b}}^{3-i+\frac{\infty}{2}} \rightarrow \mathbb{C}.$$

Its Aut-renormalized weight is $(-4, 2, 0)$. The pairing is compatible with $\text{Spin}(10)$ -action.

Proof. I define the pairing on the level of weight components. The fact that $H_{\mathfrak{a}}^{s'(\mathfrak{a})-i}[\delta, \gamma]^w$ and $H_{\mathfrak{b}}^{s'(\mathfrak{b})-j}[\delta, \gamma]^{w'}$, $\delta \in M_1^+ \sqcup M_3^+$, $\gamma \in M_1^- \sqcup M_3^-$ form two bidirect systems with a pairing in the sense of Appendix D.1 follows from Corollary 4.138. Nondegeneracy of the pairing between components as well as the degree of the pairing follows from Proposition 4.90 and Remark 4.96.

The first statement follows from Proposition D.18 and an isomorphism (6.19). Conditions of Proposition D.18 follow from Proposition 5.17 item 3.

The statement about $\text{Spin}(10)$ action follows from Proposition 4.90 item 3 and Proposition 6.20 item 2.

■

Another structure on the limiting groups is an isomorphism from the next proposition.

Proposition 6.33 *There is an isomorphism*

$$\tau^{-1}\sigma : H_{\mathfrak{b}}^{i+\frac{\infty}{2}} \cong H_{\mathfrak{a}}^{i+\frac{\infty}{2}}.$$

The isomorphisms are compatible with the grading shift described in Remark 4.96. A composition of this map with the pairing from Proposition 6.32 give a nondegenerate pairing

$$H_{\mathfrak{a}}^{i+\frac{\infty}{2}} \otimes H_{\mathfrak{a}}^{3-i+\frac{\infty}{2}} \rightarrow \mathbb{C}.$$

Its Aut-renormalized weight is $(-4, 2, 0)$. The pairing is compatible with the $\text{Spin}(10)$ -action.

This verifies Conjecture 1 item 3 for $\mathcal{X} = \mathcal{C}$.

Proof. The isomorphism is induced by (4.114). Its compatibility with the bidirect system on $H_{\mathfrak{a}}^i[\delta, \delta']$ and $H_{\mathfrak{b}}^i[\delta, \delta']$ follows from Proposition 4.142. The statement about compatibility with the shifts copies the end of the proof of Proposition 4.116. The rest follows from Proposition 6.32. ■

Corollary 6.34 *The virtual character $Z_{\mathfrak{a}}(t, q, z)$ of $\{H_{\mathfrak{a}}^{i+\frac{\infty}{2}}\}$ in the sense of Section 4.2.5 satisfies $Z_{\mathfrak{a}}(t, q, z) = -t^{-4}q^2Z_{\mathfrak{a}}(qt^{-1}, q, z^{-1})$. Equation (1.22) follows from it. The claim follows from the first item of Corollary 6.27 and Proposition 6.33.*

Finally there is a vanishing result:

Proposition 6.35 *The groups $H_{\mathfrak{c}}^{i+\frac{\infty}{2}}$, $\mathfrak{c} = \mathfrak{a}, \mathfrak{b}$ are zero for $i \neq 0, \dots, 3$.*

Proof. Follows from Proposition 4.109. ■

Remark 6.36 By Remark 4.14 and Proposition 4.48, the index $\text{ind OGr}^+(5, 10)$ is equal to 8. By (4.120), $\dim \text{OGr}^+(5, 10) = 10$. I conclude that $\text{coind OGr}^+(5, 10) + 1 = 10 - 8 + 1 = 3$. It matches with the degree of the pairing (Proposition 6.32) as promised in Conjecture 1 item 3.

In the next proposition I continue to use notations of Definition 4.63.

Proposition 6.37 Define

$$\underline{u}_i^{\mathfrak{a}} := \underline{u} \left(H_{\mathfrak{a}}^{i+\frac{\infty}{2}}(A) \right), i = 0, \dots, 3$$

for renormalized \mathbf{T} -action (Remark 4.96). It follows from Proposition 5.17 item 2 that

$$\underline{u}_0^{\mathfrak{a}} \geq -2, \underline{u}_1^{\mathfrak{a}} \geq -1, \underline{u}_2^{\mathfrak{a}} \geq -1, \underline{u}_3^{\mathfrak{a}} \geq 0.$$

The constants $\overline{u}_i^{\mathfrak{b}}$ satisfy

$$\overline{u}_0^{\mathfrak{b}} \leq -2, \overline{u}_1^{\mathfrak{b}} \leq -1, \overline{u}_2^{\mathfrak{b}} \leq -1, \overline{u}_3^{\mathfrak{b}} \leq 0.$$

Remark 6.38 The pairing in Proposition 6.33 determines a symmetry of the series coefficients $Z_{\mathfrak{a}} = \sum_{u \geq 0} c_{\mathfrak{a}, u} t^u q^u$. Constraint on u follows from Proposition 6.37. Functional equation on $Z_{\mathfrak{a}}$ Corollary 6.34 implies that $c_{\mathfrak{a}, u} = -c_{-4-\mathfrak{a}, 2+\mathfrak{a}+u}$ and $c_{\mathfrak{a}, u} = 0$ for $\mathfrak{a} < -u - 2$. This implies that $Z_{\mathfrak{a}}(t, q) \in \mathbb{Z}((t))[[q]]$. A similar argument also works for $\chi_{H_{\mathfrak{a}}^{i+\frac{\infty}{2}}}(t, q, z)$.

Explicit description of $H^{3+\frac{\infty}{2}}(A)$ Fix the algebra D generated by $\lambda^{\alpha^r}, w_{\beta^l}, \alpha^r, \beta^l \in \hat{\mathbb{E}}$ subject to commutation relations

$$\begin{aligned} [\lambda^{\alpha^r}, \lambda^{\alpha'^{r'}}] &= [w_{\beta^l}, w_{\beta'^{l'}}] = 0, \\ [\lambda^{\alpha^r}, w_{\beta^l}] &= \delta_{\beta}^{\alpha} \delta_{r+l, 0}. \end{aligned}$$

The set $\rho^{-1}(i) \subset \hat{\mathbb{E}}$ consists of two elements, which I denote by α_{\pm} . I choose in D a new set of generators:

$$\begin{aligned} \lambda_+^i &= \lambda^{\alpha_+} + \lambda^{\alpha_-}, & w_+^i &= w_{\alpha_+} + w_{\alpha_-}, \\ \lambda_-^i &= \lambda^{\alpha_+} - \lambda^{\alpha_-}, & w_-^i &= w_{\alpha_+} - w_{\alpha_-}, \\ \rho(\alpha_{\pm}) &= i. \end{aligned} \tag{6.39}$$

Note that $+$ and $-$ labelled generators mutually commute. The D -submodule U of $S_{\mathfrak{a}}[\hat{\mathbb{E}}]$ (4.70) cyclicly generated by

$$\text{vac} = \frac{1}{\lambda^{(3)^{-1}}} \frac{1}{\lambda^{(2)^{-1}}} \frac{1}{\lambda^{(1)^{-1}}} \prod_{i < 0, i \equiv 1 \pmod{2}} \lambda_-^i \prod_{i < 0} \frac{1}{\lambda_+^i}$$

can be characterized by relations

$$\begin{aligned}
\lambda_+^i vac &= 0, \quad i < 0, \\
\lambda^{3^{-1}} vac &= \lambda^{2^{-1}} vac = \lambda^{1^{-1}} vac = 0, \\
w_{\alpha^i} vac &= 0, \quad i < 0, \\
(w_i^-)^2 vac &= 0, \quad i > 0, \quad i \equiv 1 \pmod{2}, \\
w_i^- vac &= 0, \quad i > 0, \quad i \equiv 0 \pmod{2}.
\end{aligned} \tag{6.40}$$

The term "submodule" is not quite appropriate because of the infinite product in the numerator that defined vac . The defining relations (6.40) contain only finite expression in generators and could be used as the definition of the $\mathbb{C}[\hat{E}]$ -module without any references to (6.40). Here the promised elementary description of one of the semi-infinite local cohomology groups.

Proposition 6.41 *Denote by N the tensor product $A[-\infty, \infty] \otimes_P U$. There is a map*

$$H_a^{3+\frac{\infty}{2}} \cong \hat{N}, \quad P = \mathbb{C}[\hat{E}] \tag{6.42}$$

Completion \hat{N} of N is taken with respect to the system of submodules $i(-k, \delta)N \subset N$ defined by ideals $i(-k, \delta) := i[\lambda_-^i | i \leq -k] + i[\hat{E} \setminus \hat{E}^{\leq \delta}] \subset P$. Conjecturally this map is an isomorphism.

Proof. To simplify notations I denote $\hat{N}/i(-k, \delta)N = N/i(-k, \delta)N$ by $N(-k, \delta)$.

I will outline the proof first. By Proposition 4.99 the group $H_a^{3-\rho(\delta)}[\delta, \delta']$ is the tensor product $S_a[\delta, \delta'] \otimes_{\mathbb{C}[\delta, \delta']} A[\delta, \delta']$. I will fix k and δ' and construct a coherent family of maps

$$S_a[\delta, \delta'] \otimes_{\mathbb{C}[\delta, \delta']} A[\delta, \delta'] \rightarrow N(-k, \delta') \tag{6.43}$$

This will enable me to define a map

$$\lim_{\delta} H_a^{3-\rho(\delta)}[\delta, \delta'] \rightarrow \lim_{\delta} N(-k, \delta')$$

In order to define the desired map I pass to inverse limit with respect to δ' .

In the rest of the proof I will concern myself with a construction of the map (6.43).

I will use the following notations

$$U[-\infty, \delta'] := U/i[\hat{E} \setminus \hat{E}^{\leq \delta'}]U, \quad U(-k, \delta') := U[-\infty, \delta']/i[\lambda_-^i | i < -k]U[-\infty, \delta']$$

$U[-\infty, \delta']$ is a $D[-\infty, \delta']$ -module. Inclusion of Weyl algebras $D[\delta, \delta'] \rightarrow D[-\infty, \delta']$ turns $U[-\infty, \delta']$ into a $D[\delta, \delta']$ -module. I define an inclusion of $D[\delta, \delta']$ -modules $f : S_a[\delta, \delta'] \rightarrow U[-\infty, \delta']$ on the generator by

$$\varpi_a[\delta, \delta'] \rightarrow \prod_{\rho(\delta) \leq i < 0, i \equiv 1 \pmod{2}} w_i^- vac.$$

By restriction of scalars from $\mathbb{C}[-\infty, \infty]$ to $\mathbb{C}[\delta, \delta']$ $U[-\infty, \delta']$ and $U(-k, \delta')$ become $\mathbb{C}[\delta, \delta']$ -modules. By abuse of notations I denote by \mathbf{f} the composition of maps of $\mathbb{C}[\delta, \delta']$ -modules

$$S_{\mathbf{a}}[\delta, \delta'] \rightarrow U[-\infty, \delta'] \rightarrow U(-k, \delta')$$

I denote by $S_{\mathbf{a}}^{-k}[\delta, \delta']$ the image $\text{Im} \mathbf{f} \subset U(-k, \delta')$. The map \mathbf{f} induces a map of tensor products

$$A[\delta, \delta'] \underset{\mathbb{C}[\delta, \delta']}{\otimes} S_{\mathbf{a}}[\delta, \delta'] \rightarrow A[\delta, \delta'] \underset{\mathbb{C}[\delta, \delta']}{\otimes} S_{\mathbf{a}}^{-k}[\delta, \delta']$$

I denote by $S_{\mathbf{a}}(-k, \delta')$ a $\mathbb{C}[-\infty, \delta']$ -submodule in $U(-k, \delta')$ generated by $S_{\mathbf{a}}^{-k}[\delta, \delta']$.

$$\text{The action of } \mathbb{C}[-\infty, \delta'] \text{ on } S_{\mathbf{a}}(-k, \delta') \text{ factors through } Q := \mathbb{C}[\lambda_-^{-k}, \dots, \lambda_-^{\rho(\delta)}] \otimes \mathbb{C}[\delta, \delta']. \quad (6.44)$$

Moreover

$$S_{\mathbf{a}}(-k, \delta') \cong \mathbb{C}[\lambda_-^{-k}, \dots, \lambda_-^{\rho(\delta)}] \otimes S_{\mathbf{a}}^{-k}[\delta, \delta']. \quad (6.45)$$

The nontrivial part of the construction is the map

$$\mathbf{g} : A[\delta, \delta'] \underset{\mathbb{C}[\delta, \delta']}{\otimes} S_{\mathbf{a}}^{-k}[\delta, \delta'] \rightarrow A[-\infty, \delta'] \underset{\mathbb{C}[-\infty, \delta']}{\otimes} S_{\mathbf{a}}(-k, \delta') \subset A \underset{P}{\otimes} U(-k, \delta') = N(-k, \delta')$$

The map (6.43) is the composition $\mathbf{g} \circ \mathbf{f}$.

Let $\mathbf{i}^A := \mathbf{i}^A[\hat{\mathbf{E}} \setminus \hat{\mathbf{E}}^{\geq \delta}]$ be an ideal in $A[-\infty, \delta']$. Obviously $A[-\infty, \delta']/\mathbf{i}^A \cong A[\delta, \delta']$. If I manage to prove that

$$\mathbf{i}^A \underset{\mathbb{C}[-\infty, \delta']}{\otimes} S_{\mathbf{a}}(-k, \delta') = \{0\} \quad (6.46)$$

it would mean that

$$A[-\infty, \delta'] \underset{\mathbb{C}[-\infty, \delta']}{\otimes} S_{\mathbf{a}}(-k, \delta') \cong A[\delta, \delta'] \underset{Q}{\otimes} S_{\mathbf{a}}(-k, \delta') \cong A[\delta, \delta'] \underset{\mathbb{C}[\delta, \delta']}{\otimes} S_{\mathbf{a}}^{-k}[\delta, \delta']$$

I the last identification I used isomorphism (6.45). Then I would take \mathbf{g} to be the composition of the above isomorphisms.

Lemma 6.47 *The isomorphism (6.46) holds true.*

Proof. By (6.44) I can replace $A[-\infty, \delta']$ and $\mathbb{C}[-\infty, \delta']$ in $B(-k, \delta') := A[-\infty, \delta'] \underset{\mathbb{C}[-\infty, \delta']}{\otimes} S_{\mathbf{a}}(-k, \delta')$ by $A[-\infty, \delta']^{\geq -k}$ and $\mathbb{C}[-\infty, \delta']^{\geq -k}$. By definition $\lambda_+^i, i < 0$ act trivially in $S_{\mathbf{a}}(-k, \delta')$. This is why

$$\lambda^{\alpha} a = -\lambda^{\alpha'} a, a \in S_{\mathbf{a}}(-k, \delta') \quad (6.48)$$

when $\rho(\alpha) = \rho(\alpha') < 0$ and $\alpha \neq \alpha'$. By using isomorphism (6.45) any element in $B(-k, \delta')$ is a sum of the elements of the form

$$\begin{aligned} as &= a \prod_{-k \leq i \leq \rho(\delta), i \equiv 1 \pmod{2}} \lambda_{-}^i w_i^{-} s' = a \prod_{\alpha \in X} \lambda^{\alpha} t, \\ t &= \prod_{-k \leq i \leq \rho(\delta), i \equiv 1 \pmod{2}} w_i^{-} s'' \in U(-k, \delta') \\ a &\in A[-\infty, \delta']^{\geq -k}, s, s', s'' \in S_a^{-k}[\delta, \delta'] \end{aligned}$$

X is any subset \hat{E} such that $\rho : X \rightarrow [-k, \rho(\delta)] \cap (1 + 2\mathbb{Z})$ is a bijection. The last equality follows from (6.48). It is a matter of direct inspection of the diagram (2.5) to verify that for any α such that $-k \leq \rho(\alpha) \leq \rho(\delta)$ there is X as above and $\beta \in X$ such that (α, β) is a clutter. Let us pick α such that $\rho(\alpha) = -k$. Then by straightening relations (2.15)

$$\lambda^{\alpha} s = \lambda^{\alpha} \lambda^{\beta} \prod_{\beta' \in X \setminus \{\beta\}} \lambda^{\beta'} t = 0$$

It is true because if $\rho(\gamma) < -k$, then $\lambda^{\gamma} t = 0$. The same argument works when $-k < \rho(\alpha) < \rho(\delta)$. The only difference is that straightening rules and (6.48) has to be repeated several times. ■

■

7 The field - antifield pairing

The field - antifield pairing is a companion structure to the $*$ -duality pairing. Though their construction are formally distinct, the framework of the proofs in both cases is very similar. This is why I leave some of the verifications to the reader. In this section all the intervals satisfy purity condition (2.42).

7.1 The field - antifield pairing for a finite interval

Let A be an algebra based on the interval $[\delta, \delta']$. The field - antifield pairing is a pairing between cohomology of Koszul complexes $H_i(B(H_a^j(A)))$ and $H_{i'}(B(H_a^{j'}(A)))$. The pair of indices suggests that these cohomologies are pages of spectral sequences. Recall that by Proposition 4.85, $H_c^i(A)$ can be computed by using the complex $T_{\bullet}(c)$. I define $\mathcal{T}_{\bullet}(c)$ to be the diagonal complex of the Koszul bicomplex

$$\begin{aligned} \mathcal{T}_{i,j}(c) &:= B_i(T_j(c), \{\lambda^{\alpha} | \alpha \in N(c)\}), c \neq p, N(c) \text{ as in 4.51,} \\ \mathcal{T}_{i,j}(p) &:= B_i(T_j(p), \{\lambda^{\beta}(z) | \beta \in E\}). \end{aligned} \tag{7.1}$$

There are also closely related complexes $\mathcal{T}_{Kos}^\bullet(\mathfrak{c})$, the diagonal complex of

$$\begin{aligned}\mathcal{T}_{Kos}^{i,j}(\mathfrak{c}) &:= \varinjlim_n B_{-i}(K^j(A, \{(\lambda^\alpha)^n\}), \{\lambda^\beta\}), \alpha, \beta \in N(\mathfrak{c}), \mathfrak{c} \neq \mathfrak{p} \\ \mathcal{T}_{Kos}^{i,j}(\mathfrak{p}) &:= \varinjlim_n B_{-i}(K^j(A, \{(\lambda^\alpha(z))^n\}), \{\lambda^\beta(z)\}), \alpha, \beta \in E.\end{aligned}\tag{7.2}$$

It follows from Proposition 4.85 that

$$BV_{\mathfrak{c}}^i := H^i(\mathcal{T}_{Kos}(\mathfrak{c})) = H_{s(\mathfrak{c})-i}(\mathcal{T}(\mathfrak{c})).\tag{7.3}$$

The following result gives the rough idea of the size of the groups (7.3) and their relations to bi-indexed groups $H_i(B(H_a^j(A)))$.

- Proposition 7.4** 1. *The first spectral sequence of the bicomplex $\mathcal{T}_{i,j}(\mathfrak{c})$ has the first page $E_{ij}^1 := B_i(\text{Tor}_j^P(A, T_{\mathfrak{c}})) \cong B_i(H_{\mathfrak{c}}^{s(\mathfrak{c})-j}(A))$.*
2. *The second spectral sequence of the same bicomplex has the first page isomorphic to $\delta_{s(\mathfrak{c}),i} \otimes V_j$ (3.31). The rank one \mathbb{C} -linear space $\delta_{s(\mathfrak{c}),i}$ (or rank one $\mathbb{C}[z, z^{-1}]$ -modules in \mathfrak{p} -case) is concentrated in cohomological degree $s(\mathfrak{c})$.*
3. $H_i(\mathcal{T}(\mathfrak{c})) \cong V_{i-s(\mathfrak{c})}$.

Proof. An easy computation with the Künneth's formula shows that

$$H_i(B(T_{\mathfrak{c}}, \{\lambda^\alpha | \alpha \in N(\mathfrak{c})\})) = \delta_{s(\mathfrak{c}),i}.\tag{7.5}$$

The remaining proof, which uses standard spectral sequence arguments, is left to the reader. ■

I will construct a pairing

$$\begin{aligned}\mathcal{T}_i(\mathfrak{a}) \otimes \mathcal{T}_{k-16-i}(\mathfrak{a}') &\rightarrow \mathbb{C}[z, z^{-1}], \\ k &= d + |[\delta, \delta']|, d = \text{rk}(\delta, \delta') + 1\end{aligned}$$

which will descend to a field-anti-field prefect pairing on cohomology. Existence of such pairing is not surprising knowing that there is a pairing on V_i (Proposition 3.36). My construction will use three ingredients:

1. A $\mathbb{C}[z, z^{-1}]$ -linear map of complexes

$$\begin{aligned}\Upsilon : \mathcal{T}_\bullet(\mathfrak{a}) \otimes \mathcal{T}_\bullet(\mathfrak{p}) \otimes \mathcal{T}_\bullet(\mathfrak{a}') &\rightarrow \mathcal{T}_\bullet(\mathfrak{m}) \\ \Upsilon_{Kos} : \mathcal{T}_{Kos}^\bullet(\mathfrak{a}) \otimes \mathcal{T}_{Kos}^\bullet(\mathfrak{p}) \otimes \mathcal{T}_{Kos}^\bullet(\mathfrak{a}') &\rightarrow \mathcal{T}_{Kos}^\bullet(\mathfrak{m}).\end{aligned}\tag{7.6}$$

2. A d -closed $\mathbb{C}[z, z^{-1}]$ linear map

$$\text{res}_T : \mathcal{T}_k(\mathfrak{m}) \rightarrow \mathbb{C}[z, z^{-1}]. \quad (7.7)$$

3. A cocycle

$$\Theta(z) \in H_{16}(\mathcal{T}(\mathfrak{p})) \cong H^0(\mathcal{T}_{Kos}(\mathfrak{p})). \quad (7.8)$$

The pairing will be the composition

$$(a, b)_Y = \text{res}_T \circ Y(a, \Theta(z), b). \quad (7.9)$$

The map Y In my construction of Y , I use that $\mathcal{T}_\bullet(\mathfrak{c})$ is a tensor product $F_\bullet(A) \otimes_P T_\mathfrak{c} \otimes \Lambda[N(\mathfrak{c})]$. To construct Y one has to extend scalars from \mathbb{C} to $\mathbb{C}[z, z^{-1}]$ and replace P by $P[z, z^{-1}]$. The map Y (7.6) is the tensor product of three three-linear maps. The first map is the triple product of resolutions (3.39) $\cdot \times (\cdot \times \cdot)$. The second is

$$T_{\mathfrak{a}} \otimes T_{\mathfrak{p}} \otimes T_{\mathfrak{a}'} \rightarrow T_{\mathfrak{a}} \otimes_P T_{\mathfrak{p}} \otimes_P T_{\mathfrak{a}'} = T_{\mathfrak{m}}.$$

It uses an isomorphism (4.61). The third is the tautological product

$$\Lambda[N(\mathfrak{p})] \otimes \Lambda[N(\mathfrak{a})] \otimes \Lambda[N(\mathfrak{a}')] \rightarrow \Lambda[\delta, \delta'] = \Lambda[N(\mathfrak{m})].$$

The functional res_T By definition, the groups $\mathcal{T}_i(\mathfrak{m})$ are nontrivial in the range $0 \leq i \leq k$, (7.7). The top degree group

$$\begin{aligned} \mathcal{T}_k(\mathfrak{m}) &:= T_{\mathfrak{m}} \otimes \text{span} \langle \theta \otimes c \rangle, \\ \theta_X &:= \bigwedge_{\alpha \in X} \theta^\alpha, \theta := \theta_{[\delta, \delta']} \\ \text{span} \langle c \rangle &= V_d \end{aligned} \quad (7.10)$$

contains a one-dimensional cohomology group $H_k(\mathcal{T}(\mathfrak{m})) \cong \mathbb{C}$ generated by the cocycle

$$\varpi_{\mathfrak{m}} \otimes \theta \otimes c. \quad (7.11)$$

The map res_T is defined as

$$\text{res}_T : \mathcal{T}_k(\mathfrak{m}) \rightarrow \mathbb{C}[z, z^{-1}], \quad p(\lambda) \otimes \theta \otimes c \rightarrow \text{res}_{T_{\mathfrak{m}}}(p(\lambda)), \quad p(\lambda) \in T_{\mathfrak{m}}$$

is nonzero on $\varpi_{\mathfrak{m}} \otimes \theta \otimes c$. It is d -closed by trivial reasons.

7.2 The cocycle $\Theta(z)$

Let A be an algebra based on the interval $[\delta, \delta']$. The cocycle $\Theta(z)$ (7.8) is the essential part of the pairing (7.9). Its construction will depend on the choice of an element \bar{c} in cohomology of $B_\bullet(\mathring{A})$ (c.f. formula 5.19 in [2]). I have a two-step construction of $\Theta(z)$. First, I define a complex $\mathring{\mathcal{T}}_\bullet(\mathfrak{m})$ whose cohomology pairs perfectly with the cohomology of $B_\bullet(\mathring{A})$. I define Θ as cocycle in complementary cohomology group such that

$$\langle \bar{c}, \Theta \rangle = 1 \quad (7.12)$$

where \bar{c} is some distinguished element in $B_\bullet(\mathring{A})$. Second, I devise a map

$$\mathfrak{q}_* : \mathring{\mathcal{T}}_\bullet(\mathfrak{m}) \rightarrow \mathcal{T}_\bullet(\mathfrak{p})$$

which I will use to produce $\Theta(z)$:

$$\Theta(z) := \mathfrak{q}_* \Theta \in BV_{\mathfrak{p}}^0 = H^0(\mathcal{T}_{Kos}(\mathfrak{p})) = H_{16}(\mathcal{T}(\mathfrak{p})). \quad (7.13)$$

Cohomology of $B_\bullet(\mathring{A})$ I will use the notations: \mathring{P} will stand for $\mathbb{C}[(0)^0, (1)^0]$, $\mathring{\Lambda}$ for $\Lambda[(0)^0, (1)^0]$ and \mathring{D} for $D[(0)^0, (1)^0]$.

Cohomology of the Koszul complex $B(\mathring{A}) = \mathring{A} \otimes \mathring{\Lambda}$ has been studied extensively ([14],[5],[17]). By Propositions 3.36 and Corollary 2.47, cohomology $\mathring{V}_i := H_i(B(\mathring{A}))$ has a perfect pairing $\mathring{V}_i \otimes \mathring{V}_{5-i} \rightarrow \mathbb{C}$. The generator \bar{c} of \mathring{V}_5 is represented by the cocycle

$$\Gamma_{\beta\beta}^m \lambda^\beta \theta^\beta \Gamma_{\gamma\delta}^n \lambda^\gamma \theta^\delta \Gamma_{\epsilon\epsilon}^k \lambda^\epsilon \theta^\epsilon \Gamma_{\mu\nu}^{mnk} \theta^\mu \theta^\nu.$$

By Proposition 3.36, linear spaces \mathring{V}_i are the generating spaces of the minimal free \mathring{P} -resolution $F_\bullet(\mathring{A})$ of \mathring{A} .

The pairing $H_{21-i}(\mathring{\mathcal{T}}(\mathfrak{m})) \otimes \mathring{V}_i \rightarrow \mathbb{C}$. Here is the notations that I will use for some \mathring{D} -modules

$$\mathring{T}_{\mathfrak{m}} := T_{\mathfrak{m}}[(0)^0, (1)^0], \quad \mathring{T}_{(0)} := \mathring{P}$$

and for some complexes

$$\begin{aligned} \mathring{\mathcal{T}}_\bullet(0) &:= B_\bullet(F_\bullet(\mathring{A})), \\ \mathring{\mathcal{T}}_\bullet(\mathfrak{m}) &:= B_\bullet(\mathring{\mathcal{T}}_\bullet(\mathfrak{m})), \quad \mathring{\mathcal{T}}_\bullet(\mathfrak{m}) := F_\bullet(\mathring{A}) \otimes_{\mathring{P}} \mathring{T}_{\mathfrak{m}}. \end{aligned}$$

The pairing that I am about to construct will be used in (7.12) for definition of Θ .

Proposition 7.14 1. Cohomology of $\mathring{\mathcal{T}}_{\bullet}(0)$ coincide with cohomology of $B(\mathring{A})$.

2. There is a nondegenerate pairing $H_{21-i}(\mathring{\mathcal{T}}(\mathfrak{m})) \otimes H_i(B(\mathring{A})) \rightarrow \mathbb{C}$.

Proof. The first statement is obvious.

The complex $F_{\bullet}(\mathring{A})$ has length 5. This is why

$$\mathring{\mathcal{T}}_i(\mathfrak{c}) = \bigoplus_{k+l=i} \mathring{\mathcal{T}}_k(\mathfrak{c}) \otimes \mathring{\Lambda}^l, \mathfrak{c} = (0), \mathfrak{m} \quad (7.15)$$

have length 21. It is easy to construct a nondegenerate pairing between complexes

$$\mathring{\mathcal{T}}_{21-i}(\mathfrak{m}) \otimes \mathring{\mathcal{T}}_i(0) \rightarrow \mathbb{C}. \quad (7.16)$$

The complex 7.15 is a tensor product. The $\mathring{\mathcal{T}}$ -tensor factors are equipped with the pairing from Lemma 4.93. Exterior algebra has the inner product

$$\mathring{\Lambda}^i \otimes \mathring{\Lambda}^{16-i} \rightarrow \mathring{\Lambda}^{16} \cong \mathbb{C}.$$

The tensor product of the last two inner products define (7.16). The pairing that I just constructed induces the pairing in cohomologies stated in the second item of the proposition. ■

Definition 7.17 I define Θ to be a nontrivial cocycle in $\mathring{\mathcal{T}}_{16}(\mathfrak{m})$ that satisfies (7.12).

Definition of the map \mathfrak{q}_* The map

$$\begin{aligned} \mathfrak{q} : \mathring{P} &\rightarrow P[z, z^{-1}] \\ \mathfrak{q}(\lambda^{\beta}) &:= \lambda_{[\delta, \delta']}^{\beta}(z) \end{aligned} \quad (7.18)$$

descends to a homomorphism of algebras

$$\mathring{A} \rightarrow A[z, z^{-1}]. \quad (7.19)$$

\mathfrak{q} induces a map of complexes $\mathring{\mathcal{T}}_{\bullet}(\mathfrak{m}) \rightarrow \mathcal{T}_{\bullet}(\mathfrak{p})$. Here are the details. The minimal free resolution $F_{\bullet}(A)[z, z^{-1}]$ of $A[z, z^{-1}]$ is a module over $P[z, z^{-1}]$. After restriction of scalars via \mathfrak{q} it becomes a complex of $\mathring{P}[z, z^{-1}]$ -modules. Freeness of the resolution allows to lift the map (7.19) to a map of the complexes

$$F_{\bullet}(\mathring{A})[z, z^{-1}] \rightarrow F_{\bullet}(A)[z, z^{-1}] \quad (7.20)$$

of the $\mathring{P}[z, z^{-1}]$ -modules.

The map 7.18 induces a map of $\hat{P}[z, z^{-1}]$ -modules:

$$\begin{aligned} \hat{T}(\mathfrak{m})[z, z^{-1}] &\rightarrow T(\mathfrak{p}) \\ \prod_{\beta \in E} (\lambda^\beta)^{-1-k_\beta} &\rightarrow \prod_{\beta \in E} (\lambda^\beta(z))^{-1-k_\beta} \prod_{\alpha \in N(\mathfrak{a}) \cup N(\mathfrak{a}')} (\lambda^\alpha)^{-1}. \end{aligned} \quad (7.21)$$

I define the map of complexes

$$\mathfrak{q}_* : \hat{\mathcal{T}}_\bullet(\mathfrak{m}) = F(\hat{A}) \otimes_{\hat{P}} \hat{T}(\mathfrak{m}) \otimes \hat{\Lambda} \rightarrow F(A) \otimes_{P[z, z^{-1}]} T(\mathfrak{p}) \otimes \hat{\Lambda} = \mathcal{T}_\bullet(\mathfrak{p})$$

as a tensor product of (7.20), (7.21) and the identity map on $\hat{\Lambda}$. I define $\Theta(z)$ by the formula (7.13). Before we can move on we have to check nontrivially of $\Theta(z)$.

Proposition 7.22

$$\Theta(z) \neq 0 \in H_{16}(\mathcal{T}(\mathfrak{p})).$$

Proof. Composition of (7.19) with projection $\mathfrak{p} : A[z, z^{-1}] \rightarrow \hat{A}[z, z^{-1}]$ is the identity on $\hat{A}[z, z^{-1}]$. I use this to define a map $\mathfrak{p}_* : \mathcal{T}_\bullet(\mathfrak{p}) \rightarrow \hat{\mathcal{T}}_\bullet(\mathfrak{m})$ such that $\mathfrak{p}_* \circ \mathfrak{q}_*$ is homotopic to identity. From this I conclude that $\mathfrak{q}_*\Theta \neq 0$ ■

7.3 Nondegeneracy of the field-antifield pairing

Now we are in position to put together all the ingredients, define $(a, b)_Y$ and verify nondegeneracy of $(a, b)_Y$. In the notations (2.28) I define

$$\begin{aligned} u(\mathfrak{m}) &:= \sum_{\alpha \in N(\mathfrak{m})} u(\alpha) \\ r'(\mathfrak{m}) \Big| z^{r'(\mathfrak{m})} &:= \prod_{\alpha \in N(\mathfrak{m})} v_\alpha(z). \end{aligned} \quad (7.23)$$

where N is defined in (4.51). Here is the main statement of this section.

Proposition 7.24 1. *Bilinear form (7.9) defines nondegenerate pairings*

$$\begin{aligned} H_i(\mathcal{T}(\mathfrak{a})) \otimes H_{k-16-i}(\mathcal{T}(\mathfrak{a}')) &\xrightarrow{(a, b)_Y} \mathbb{C}[z, z^{-1}] \\ BV_{\mathfrak{a}}^j \otimes BV_{\mathfrak{a}'}^{k'-j} &\xrightarrow{(a, b)_Y} \mathbb{C}[z, z^{-1}] \\ k = s(\mathfrak{m}) + s'(\mathfrak{m}), \quad k' &= s'(\mathfrak{m}) - s(\mathfrak{m}). \end{aligned} \quad (7.25)$$

2. *The non-renormalized Aut-weight of the pairing is equal to*

$$(a[\delta, \delta'] - s(\mathfrak{m}), u[\delta, \delta'] - u(\mathfrak{m}), r[\delta, \delta'] - r(\mathfrak{m})) \quad (\text{see 4.59, 4.45 for notations}).$$

Proof. The two pairings are related by the isomorphism (7.3). The spectral sequence from item 2, Proposition 7.4 is induced by the filtration

$$F_j(\mathcal{T}_i(\mathfrak{c})) = \bigoplus_{k+l=i, k \leq j} T_k(\mathfrak{c}) \otimes \Lambda^l, \mathfrak{c} \neq \mathfrak{p},$$

$$F_j(\mathcal{T}_i(\mathfrak{p})) = \bigoplus_{k+l=i, k \leq j} T_k(\mathfrak{p}) \otimes \mathring{\Lambda}^l.$$

Filtrations $F_j(\mathcal{T}_i(\mathfrak{p})), F_j(\mathcal{T}_i(\mathfrak{a})), F_j(\mathcal{T}_i(\mathfrak{a}'))$ are compatible with the triple product (7.6). That is why Υ induces a map of the first pages:

$$(\delta_{s(\mathfrak{a}), i} \otimes V_j) \otimes (\delta_{s(\mathfrak{p}), i'} \otimes V_{j'}) \otimes (\delta_{s(\mathfrak{a}'), i''} \otimes V_{j''}) \rightarrow \delta_{s(\mathfrak{m}), i+i'+i''} \otimes V_{j+j'+j''}. \quad (7.26)$$

Only one group in the sum $\bigoplus_{i+j=16} \delta_{s(\mathfrak{p}), i} \otimes V_j$ is nontrivial. It is $\delta_{s(\mathfrak{p}), 16} \otimes V_0 \cong \mathbb{C}$. The top component of

$$\Theta(z) = \sum_{i=0}^{16} \Theta_i(z), \Theta_i(z) \in T_{16-i}(\mathfrak{p}) \otimes \mathring{\Lambda}^i$$

with respect to filtration must be equal to (up to a nonzero factor)

$$\Theta_{16}(z) = \varpi_{\mathfrak{p}} \theta \in T_{\mathfrak{p}} \otimes \mathring{\Lambda}^{16} = T_0(\mathfrak{p}) \otimes \mathring{\Lambda}^{16}. \quad (7.27)$$

The inner product (7.9) induces a pairing between first pages of spectral sequences. If I prove that this derived pairing is not degenerate, then it will automatically imply that (7.9) is not degenerate.

To define the pairing between first pages, I use the same scheme as for (7.9) but replace (7.6) by (7.26), (7.13) by (7.27). The resulting pairing

$$\varpi_{\mathfrak{a}} \theta_{N(\mathfrak{a})} \otimes V_{d-j} \otimes \varpi_{\mathfrak{p}} \theta_{\mathfrak{E}} \otimes V_0 \otimes \varpi_{\mathfrak{a}'} \theta_{N(\mathfrak{a}')} \otimes V_j \rightarrow \varpi_{\mathfrak{m}} \theta_{[\delta, \delta']} \otimes V_d \cong \mathbb{C}$$

is not degenerate because it is constructed from nondegenerate pairing on $\bigoplus_i V_i$ (Proposition 3.36) and because $\varpi_{\mathfrak{a}} \varpi_{\mathfrak{p}} \varpi_{\mathfrak{a}'} = \varpi_{\mathfrak{m}}$ (see the proof of Proposition 4.60).

Lemma 7.28 *The Aut-weight of $\Theta(z)$ is equal to $(0, 0, 0)$.*

Proof. The easiest way to determine this is to observe that the spectral sequence from Proposition 7.24 is compatible with *Aut*-action. Thus it is safe to compute the weight of $\Theta(z)$ by calculating the weight of its leading component $\Theta_{16}(z)$ (7.27) for which the answer is obvious. ■

Lemma 7.28 implies that $\Theta(z)$ doesn't contribute to the *Aut*-weight of the pairing. I conclude that the weight in question is equal to the weight of the element (7.11), which is the sum of $(-a[\delta, \delta'], -u[\delta, \delta'], -r[\delta, \delta'])$ -the weight of $\varpi_{\mathfrak{m}} \otimes c$ (see (4.45) and Proposition 4.90) and of

$$(s[\delta, \delta'], u[\delta, \delta'], r'[\delta, \delta']),$$

the weight of $\theta_{[\delta, \delta']}$. ■

Remark 7.29 (c.f. 4.96) I define a renormalized Aut -action on $\mathcal{T}_\bullet(\mathfrak{a})$ by twisting on the character $\chi'[\delta, (1)^{-1}]^{-1} = \chi^{-1}[\delta, (1)^{-1}] \det \mathbb{C}[\delta, (1)^{-1}]$ (Proposition 4.44, equation (4.37)) and $\mathcal{T}_\bullet(\mathfrak{a}')$ by $\chi'[(0)^1, \delta']^{-1}$

Remark 7.30 Proposition 7.4 implies equality of virtual characters

$$ZBV_{\mathfrak{c}}^{bare}(t, q, z) := \sum_i (-1)^i \chi_{BV_{\mathfrak{c}}^i}(t, q, z) = Z_{\mathfrak{c}}^{bare}(t, q, z) \Lambda S_+(-t, q, z). \quad (7.31)$$

The pairing (7.25) implies equality of the virtual characters:

$$\begin{aligned} ZBV_{\mathfrak{a}}^{bare}(t, q, z) &= \\ &= (-1)^{s'(\mathfrak{m})-s(\mathfrak{m})} t^{s(\mathfrak{m})-a(\mathfrak{m})} q^{u(\mathfrak{m})-u(\mathfrak{m})} z^{r'(\mathfrak{m})-r(\mathfrak{m})} ZBV_{\mathfrak{a}'}^{bare}(t^{-1}, q^{-1}, z^{-1}). \end{aligned} \quad (7.32)$$

$$ZBV_{\mathfrak{a}}^{bare}[\delta, \delta'](t, q, z) = ZBV_{\mathfrak{a}'}^{bare}[\sigma(\delta'), \sigma(\delta)](t, q^{-1}, Sz)$$

the last equality is induced by the isomorphism σ (2.12).

Equality

$$\Lambda S_+(t, q, z) = t^{s(\mathfrak{m})} q^{u(\mathfrak{m})} z^{r'(\mathfrak{m})} \Lambda S_+(t^{-1}, q^{-1}, z^{-1})$$

comes from the pairing

$$\Lambda^i S_+ \otimes \Lambda^{|\delta, \delta'| - i} S_+ \rightarrow \Lambda^{|\delta, \delta'|} S_+.$$

Proposition 7.33 The virtual character $Z_{\mathfrak{a}}[\delta, \delta']$ (4.110) satisfies

$$\begin{aligned} Z_{\mathfrak{a}}[\delta, \delta'](t, q, z) &= \\ &= -t^{-8} Z_{\mathfrak{a}}[\sigma(\delta'), \sigma(\delta)](t^{-1}, q, Sz^{-1}). \end{aligned} \quad (7.34)$$

Proof. From (7.31) and (7.32), I deduce that

$$\begin{aligned} Z_{\mathfrak{a}}^{bare}(t, q, z) &= (-1)^{s'(\mathfrak{m})} t^{-a(\mathfrak{m})} q^{-u(\mathfrak{m})} z^{-r(\mathfrak{m})} \times \\ &\times Z_{\mathfrak{a}'}^{bare}(t^{-1}, q^{-1}, z^{-1}). \end{aligned}$$

This implies equation (7.34) for $Z_{\mathfrak{a}}[\delta, \delta']$. Derivation of (7.34) uses formula (4.117), items (2, 4) from Proposition 4.44 and equation (2.44), which summarize to

$$a(\mathfrak{m}) = a(\mathfrak{a}) + a(\mathfrak{a}'),$$

$$u(\mathfrak{m}) = u(\mathfrak{a}) + u(\mathfrak{a}'),$$

$$r(\mathfrak{m}) = r(\mathfrak{a}) + r(\mathfrak{a}').$$

■

Equation (7.34) simplifies if I set $[\delta, \delta'] = [(0)^{-N}, (1)^N]$. Denote $Z_{\mathbf{a}}[(0)^{-N}, (1)^N]$ by $Z'[N]$. Then (7.34) becomes (7.43). Operator S disappears in Sz^{-1} because representation of $\tilde{\mathbf{T}}^5$ in $H_{\mathbf{a}}^i(A[(0)^{-N}, (1)^N])$ is a restriction of $\text{Spin}(10, \mathbb{C})$ -representation and $S(z) = SzS^{-1}, S \in \text{Spin}(10, \mathbb{C})$.

7.4 The definition of $BV^{i+\frac{\infty}{2}}(\mathbf{c})$

Exposition of this section follows closely Section 6.3.

Fix $[\delta_2, \delta_4] \subset [\delta_1, \delta_4]$. The map $\text{th}_{\mathbf{p}}$, corresponding to $\mathbf{p} : A[\delta_1, \delta_4] \rightarrow A[\delta_2, \delta_4]$ induces the map of complexes

$$\begin{aligned} \mathcal{T}_{Kos}^{\bullet}(\mathbf{a})[\delta_2, \delta_4] &\rightarrow \mathcal{T}_{Kos}^{\bullet}(\mathbf{a})[\delta_1, \delta_4][\text{codim}], \\ \text{codim} &= s'[\delta_1, \delta_4] - s'[\delta_2, \delta_4]. \end{aligned}$$

I modify the map by multiplying the image of th on $\theta_{[\delta_1, \delta_4] \setminus [\delta_2, \delta_4]}$ (7.10). As a result, I get the map

$$\begin{aligned} \text{th}'_{\mathbf{p}} : \mathcal{T}_{Kos}^{\bullet}(\mathbf{a})[\delta_2, \delta_4] &\rightarrow \mathcal{T}_{Kos}^{\bullet}(\mathbf{a})[\delta_1, \delta_4][\text{hcodim}], \\ \text{hcodim} &= s'[\delta_1, \delta_4] - s'[\delta_2, \delta_4] - (s[\delta_1, \delta_4] - s[\delta_2, \delta_4]). \end{aligned} \tag{7.35}$$

I leave to the reader verification that $\mathcal{T}_{Kos}^{\bullet}(\mathbf{a})$ (7.2) and $\mathcal{T}_P^k(\mathbf{a}, M), M = A$ from (3.20) compute the same cohomology. I also won't prove a simple statement that map (7.35) coincides with (3.21) when $R = A[\delta_1, \delta_4], S = A[\delta_2, \delta_4]$.

Proposition 7.36 Fix $\delta_1 < \delta_2 < (0)^{-1}, (1)^{-1} < \delta_3 < \delta_4$ and $\mathbf{c} = \mathbf{a}, \mathbf{a}', \mathbf{p}, \mathbf{m}, \mathbf{b}$.

There is a commutative diagram

$$\begin{array}{ccc} BV_{\mathbf{c}}^i[\delta_2, \delta_4] & \xrightarrow{\mathbf{p}'} & BV_{\mathbf{c}}^i[\delta_2, \delta_3] \\ \downarrow \text{th}'_{\mathbf{q}} & & \downarrow \text{th}'_{\mathbf{q}'} \\ BV_{\mathbf{c}}^{i+\text{hcodim}}[\delta_1, \delta_4] & \xrightarrow{\mathbf{p}} & BV_{\mathbf{c}}^{i+\text{hcodim}}[\delta_1, \delta_3]. \end{array} \tag{7.37}$$

The maps are induced from the commutative diagram of algebras (4.126) and the maps $\text{th}'_{\mathbf{q}}, \text{th}'_{\mathbf{q}'}$ (7.35).

$$\text{hcodim} = \rho(\delta_2) - \rho(\delta_1) - |[\delta_1, (1)^{-1}] \setminus [\delta_2, (1)^{-1}]|.$$

Proof. The proof is similar to the proof of Proposition 4.124 and will be omitted. ■

Here is formal definition of the limits group.

$$BV_{\mathbf{c}}^{i+\frac{\infty}{2}}(A) := \bigoplus_w \limlim_{\substack{\delta \rightarrow \delta'}} BV_{\mathbf{c}}^{i+\text{hd}(\mathbf{c})}[\delta, \delta']^w, \mathbf{c} = \mathbf{a}, \mathbf{a}', \text{hd}(\mathbf{c}) = s'(\mathbf{c}) - s(\mathbf{c}) \tag{7.38}$$

where the structure maps come from the diagram (7.37) and the function s' from (4.87). I give only statements of the next three proposition. The interested reader will recover the proofs without a difficulty if he will use the proofs of the analogous statements from Section 6.3 as a template.

Proposition 7.39 *Suppose $[\delta_2, \delta_4] \subset [\delta_1, \delta_4]$ and $\delta_2 < (1)^{-1}, (0)^0 < \delta_4$, and that $A[\delta_i, \delta_4], i = 1, 2$ are Gorenstein. Then there are commutative diagrams*

$$\begin{array}{ccc}
BV_{\mathbf{a}}^i[\delta_2, \delta_4] \otimes BV_{\mathbf{p}}^j[\delta_1, \delta_4] \otimes BV_{\mathbf{a}'}^k[\delta_1, \delta_4] & \xrightarrow{\Upsilon_{Kos} \circ (\text{id} \otimes \mathbf{p} \otimes \mathbf{p})} & BV_{\mathbf{m}}^{i+j+k}[\delta_2, \delta_4] \\
\downarrow \text{th}_{\mathbf{p}} \otimes \text{id} \otimes \text{id} & & \downarrow \text{th}_{\mathbf{p}} \\
BV_{\mathbf{a}}^{i+\text{hcodim}}[\delta_1, \delta_4] \otimes BV_{\mathbf{p}}^j[\delta_1, \delta_4] \otimes BV_{\mathbf{a}'}^k[\delta_1, \delta_4] & \xrightarrow{\Upsilon_{Kos}} & BV_{\mathbf{m}}^{i+j+k+\text{hcodim}}[\delta_1, \delta_4] \\
BV_{\mathbf{a}}^i[\delta_1, \delta_4] \otimes BV_{\mathbf{p}}^j[\delta_1, \delta_4] \otimes BV_{\mathbf{a}'}^k[\delta_2, \delta_4] & \xrightarrow{\Upsilon_{Kos} \circ (\mathbf{p} \otimes \mathbf{p} \otimes \text{id})} & BV_{\mathbf{m}}^{i+j+k}[\delta_2, \delta_4] \\
\downarrow \text{id} \otimes \text{id} \otimes \text{th}_{\mathbf{p}} & & \downarrow \text{th}_{\mathbf{p}} \\
BV_{\mathbf{a}}^i[\delta_1, \delta_4] \otimes BV_{\mathbf{p}}^j[\delta_1, \delta_4] \otimes BV_{\mathbf{a}'}^{k+\text{hcodim}}[\delta_1, \delta_4] & \xrightarrow{\Upsilon_{Kos}} & BV_{\mathbf{m}}^{i+j+k+\text{hcodim}}[\delta_1, \delta_4]
\end{array}$$

where \mathbf{p} is the map induced by the restriction homomorphism $A[\delta_1, \delta_4] \rightarrow A[\delta_2, \delta_4]$, $\text{th}_{\mathbf{p}}$ is the modified Thom map (7.35), Υ_{Kos} is the product map (7.6).

Proposition 7.40 *In the assumptions of Proposition 7.39, the pairing (7.9) satisfies*

$$(\text{th}'_{\mathbf{q}}(a), b)_{\Upsilon} = (a, q(b))_{\Upsilon}$$

where $a \in BV_{\mathbf{a}}^i[\delta_2, \delta_4], b \in BV_{\mathbf{a}'}^j[\delta_1, \delta_4]$, $i + j + \rho(\delta_2) - \rho(\delta_1) - (|\delta_1, \delta_4| - |\delta_2, \delta_4|) = \text{rk}(\delta_1, \delta_4) + 1 - |\delta_1, \delta_4|$.

There is an isomorphism $\sigma : BV_{\mathbf{a}}^{i+\frac{\infty}{2}}(A) \rightarrow BV_{\mathbf{a}'}^{i+\frac{\infty}{2}}(A)$ induced by σ (2.12) The above information enables me to define the pairing

$$BV_{\mathbf{a}}^{i+\frac{\infty}{2}}(A) \otimes BV_{\mathbf{a}'}^{3-i+\frac{\infty}{2}}(A) \rightarrow \mathbb{C}. \quad (7.41)$$

Corollary 7.42

$$Z_{\mathbf{a}}(t, q, z) = -t^{-8} Z_{\mathbf{a}}(t^{-1}, q, z^{-1}). \quad (7.43)$$

Remark 7.44 *Spectral sequence arguments shows that*

$$ZBV_{\mathbf{a}}(t, q) = \sum_{u \geq 0, a} b_{a,u} t^a q^u = Z_{\mathbf{a}}(t, q) Z_S(t, q), \quad Z_S(t, q) = \prod_{u > 0} (1 - t^{-1} q^u)^{16} \prod_{u \geq 0} (1 - t q^u)^{16}$$

$Z_S(t, q) \in \mathbb{Z}[t, t^{-1}][[q]]$ is the renormalized virtual character of spinors. The above equality and Remark 6.38 implies that $ZBV_{\mathbf{a}}(t, q) \in \mathbb{Z}((t))[[q]]$. The symmetry of the coefficients $b_{a,u} = -b_{-8-a,u}$ of $ZBV_{\mathbf{a}}(t, q)$ that is induced by the pairing 7.41 implies that in fact $ZBV_{\mathbf{a}}(t, q) \in \mathbb{Z}[t, t^{-1}][[q]]$. From this I conclude that $Z_{\mathbf{a}}(t, q) \in \mathbb{Z}((t))[[q]] \cap \mathbb{Q}(t)[[q]]$.

8 $H_{\mathfrak{a}}^i(A)$ and localization

Let A be the algebra based on $[\delta, \delta']$. In addition, all the intervals in this section satisfy (2.42). In this section, I will address the problem of denominators outlined in the introduction.

The linear space $H_{\mathfrak{a}}^i(A)$ is by construction a P -module. In this section, I assume that $[(0)^0, (1)^0] \subset [\delta, \delta']$. The linear space $H_{\mathfrak{a}}^i(A)$ becomes \mathring{P} -module. In practice, it is often convenient to localize elements of $H_{\mathfrak{a}}^i(A)$ by inverting some of the generators $\lambda^{\mathfrak{B}} := \lambda^{\mathfrak{B}^0}$ without significantly changing the size of group $H_{\mathfrak{a}}^i(A)$. A possible way to do this is in the formalism of the Čech complex.

Recall that the extended Čech complex of an R -module M is the complex

$$M \rightarrow \prod_{i_0} M_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow M_{f_1 \dots f_r} =: \check{\text{Cech}}_e^{\bullet}(\mathfrak{U}, M)$$

where $M_{f_1 \dots f_r}$ is a localization with respect to $f_1 \dots f_r \in R$. \mathfrak{U} is the open cover of $\text{Spec } R$ by $U_i = \{x \in \text{Spec } R \mid f_i(x) \neq 0\}$. A chain from $\check{\text{Cech}}_e^m(\mathfrak{U}, \mathcal{F})$ is a family of elements $\phi = \{\phi_{\alpha_0, \dots, \alpha_m}\} \in \bigoplus M_{f_{\alpha_0} \dots f_{\alpha_m}}$. The differential is defined by the formula

$$(d\phi)_{\alpha_0 \dots \alpha_{m+1}} = \sum_{i=0}^{m+1} (-1)^i \text{res}(\phi_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{m+1}}).$$

There is a short exact sequence of complexes

$$0 \rightarrow \check{\text{Cech}}_e^{\bullet}(\mathfrak{U}, M) \rightarrow \check{\text{Cech}}_e^{\bullet}(\mathfrak{U}, M) \rightarrow M \rightarrow 0 \quad (8.1)$$

where $\check{\text{Cech}}_e^{\bullet}(\mathfrak{U}, M)$ is the classical Čech complex.

In our application I fix an open covering $\mathfrak{U} = \{U_{\alpha} \mid \lambda^{\alpha} \neq 0, \alpha \in E\}$ of $\mathbb{C}^{16} \setminus \{0\} \subset \mathbb{C}^{16}$. I use order

$$\begin{aligned} \dots < (4)^{r-1} < (0)^r < (3)^{r-1} < (12)^r < (2)^{r-1} < (13)^r < (1)^{r-1} < (14)^r < (23)^r < \\ < (15)^r < (24)^r < (25)^r < (34)^r < (35)^r < (5)^r < (45)^r < (4)^r < (0)^{r+1} < \dots \end{aligned} \quad (8.2)$$

on $E \subset \hat{E}$ to order indices. The algebra R in our case is \mathring{P} and sequence $\{f_i\}$ is $\{\lambda^{\alpha}\}$, M is some \mathring{P} -module.

$\check{\text{Cech}}_{e, \leq n}^{\bullet}(\mathfrak{U}, M)$ is a sub complex of $\check{\text{Cech}}_e^{\bullet}(\mathfrak{U}, M)$ spanned by chains whose denominators have degrees $\leq n$. It is explained in the Appendix A4.1 [21] that the complex $\check{\text{Cech}}_{e, \leq n}^{\bullet}(\mathfrak{U}, M)$ is isomorphic to Koszul complex $K(M, \{\lambda^{\mathfrak{B}}\}, n)$ and

$$\check{\text{Cech}}_e^{\bullet}(\mathfrak{U}, M) \cong \varinjlim_n K(M, \{\lambda^{\mathfrak{B}}\}, n). \quad (8.3)$$

After I apply to exact sequence (8.1) with $M = A[\delta, \delta']$ the functor $\varinjlim K(?, [\delta, (1)^{-1}], n)$ and use identification (8.3), I will get the long exact sequence

$$0 \rightarrow H^0(C) \rightarrow H_{\mathfrak{a}'}^0(A) \rightarrow H_{\mathfrak{a}}^0(A) \rightarrow H^1(C) \rightarrow \dots \quad (8.4)$$

The ideal \mathfrak{a}' is equal $i[\delta, (1)^0]$, C^\bullet is the total complex of the bicomplex $\varinjlim K^\bullet(\check{\text{Cech}}^\bullet(\mathfrak{U}, A), [\delta, (1)^{-1}], n)$.

The cohomology of C^\bullet have a simple structure.

Proposition 8.5

$$H^{i+1}(C) = H_{\mathfrak{a}}^i(A) \text{ for } -\rho(\delta) \leq i \leq -\rho(\delta) + 3$$

and

$$H^i(C) = H_{\mathfrak{a}'}^i(A) \text{ for } -\rho(\delta) + 8 \leq i \leq -\rho(\delta) + 11.$$

For other values of i $H^i(C) = 0$.

There is also an obvious modifications of these isomorphisms when $\delta \rightarrow -\infty, \delta' \rightarrow \infty$.

Proof. I know (Proposition 4.109) that $H_{\mathfrak{a}}^i(A) \neq 0$ for $-\rho(\delta) \leq i \leq -\rho(\delta) + 3$ and $H_{\mathfrak{a}'}^i(A) \neq 0$ for $-\rho(\delta) + 8 \leq i \leq -\rho(\delta) + 11$ (use (4.5) and Proposition 4.109). The result follows from the exact sequence (8.4). ■

Let \widetilde{M} be the coherent sheaf on \mathbb{C}^{16} associated with the module M . I will use the same notation for the restriction of \widetilde{M} on $\mathbb{C}^{16} \setminus \{0\}$. I denote the Čech cohomology $H^i \check{\text{Cech}}^\bullet(\mathfrak{U}, M)$ of \widetilde{M} by $H^i(\mathbb{C}^{16} \setminus \{0\}, \widetilde{M})$. With the module $M = H_{\mathfrak{a}}^i(A)$ I associate the coherent sheaf $\widetilde{H}_{\mathfrak{a}}^i(A)$ on $\mathbb{C}^{16} \setminus \{0\}$.

Proposition 8.6 1. *There is a spectral sequence*

$$H^i(\mathbb{C}^{16} \setminus \{0\}, \widetilde{H}_{\mathfrak{a}}^j(A)) \Rightarrow H^{i+j+1}(C). \quad (8.7)$$

2. *There is a similar spectral sequence*

$$H^i(\mathbb{C}^{16} \setminus \{0\}, \widetilde{H}_{\mathfrak{a}}^{j+\frac{\infty}{2}}(A)) \Rightarrow H^{i+j+1+\frac{\infty}{2}}(C[-\infty, \infty]).$$

Proof. Decomposition $[\delta, (1)^0] = [\delta, (1)^{-1}] \cup [(0)^0, (1)^0]$ enables me to apply Construction A.12 to Koszul complex $K(A, [\delta, (1)^0], n)$. By the above discussion the corresponding spectral sequence is isomorphic to (8.7).

The second statement follows from the first because th is a homomorphism of \mathring{P} -modules. ■

9 Spaces of formal maps

There is an informal way to interpret elements of the space $Z_{\mathcal{X}}^{poly}$ as probes of the target \mathcal{X} . Elements of $Z_{\mathcal{X}}^{poly}$, which are represented by Laurent polynomials, are the true maps from \mathbb{C}^\times to \mathcal{X} . Alternatively, one can take maps from the infinitesimal punctured disk to \mathcal{X} as the probes. Intuitively it is clear that infinitesimal maps do not penetrate far into \mathcal{X} . In particular, they should be less sensitive to the fundamental group of \mathcal{X} than Laurent maps. In this section I propose some conjectures about the relations of the groups $H^{i+\frac{\infty}{2}}$ based on spaces of maps of these two types.

I formalize these descriptions of these spaces by using the language of algebra. There are three versions of the algebra $A[\delta, \delta']$ when $\delta \rightarrow -\infty, \delta' \rightarrow \infty$, which I will use in this section. My exposition is less formal than in previous sections. The reason is that this section contains mostly conjectures.

Inclusions of intervals $[\delta, \delta'] \subset [\gamma, \gamma']$ define an embedding $\mathbb{C}[\delta, \delta'] \rightarrow \mathbb{C}[\gamma, \gamma']$ which is tautological on the generators. I define $\mathbb{C}(-\infty, \infty) = \mathbb{C}[\hat{E}]$ to be the direct limit $\lim_{\delta, \delta' \rightarrow} \mathbb{C}[\delta, \delta']$. The polynomial algebra $\mathbb{C}[\hat{E}]$ have three particularly interesting linear topologies. The bases of these topologies are generated by the system of ideals:

1. $i_{\pm}(n) = i((-\infty, (1)^{-n}] \cup [(0)^n, \infty)) \subset,$
2. $i_+(n) = i([(1)^n, \infty),$
3. $i_-(n) = i((-\infty, (1)^{-n})).$

By construction $i_{\pm}(n+1) \subset i_{\pm}(n), i_+(n+1) \subset i_+(n), i_-(n+1) \subset i_-(n)$. Define the completions

$$\begin{aligned} \mathbb{C}[-\infty, \infty] &= \varprojlim_n \mathbb{C}(-\infty, \infty) / i_{\pm}(n), \\ \mathbb{C}(-\infty, \infty] &= \varprojlim_n \mathbb{C}(-\infty, \infty) / i_+(n), \\ \mathbb{C}[-\infty, \infty) &= \varprojlim_n \mathbb{C}(-\infty, \infty) / i_-(n). \end{aligned} \tag{9.1}$$

Maps which are identity on the generators defines continuous homomorphisms

$$m_+ : \mathbb{C}(-\infty, \infty] \rightarrow \mathbb{C}[-\infty, \infty], \quad m_- : \mathbb{C}[-\infty, \infty) \rightarrow \mathbb{C}[-\infty, \infty].$$

Relations Γ^{s^k} (2.2) ($N = -\infty, N' = \infty$) are the elements of all the three algebras (9.1). The ideals $\mathfrak{r} \subset \mathbb{C}[-\infty, \infty], \mathfrak{q}_+ \subset \mathbb{C}(-\infty, \infty], \mathfrak{q}_- \subset \mathbb{C}[-\infty, \infty)$ are the topological closure of the ideals generated by Γ^{s^k} . Introduce the algebras

$$A[-\infty, \infty] = \mathbb{C}[-\infty, \infty] / \mathfrak{r},$$

$$A(-\infty, \infty] = \mathbb{C}[-\infty, \infty)/\mathfrak{q}_+,$$

$$A[-\infty, \infty) = \mathbb{C}[-\infty, \infty)/\mathfrak{q}_-.$$

The homomorphisms m_+ , m_- define the maps

$$m_+ : A(-\infty, \infty] \rightarrow A[-\infty, \infty], \quad m_- : A[-\infty, \infty) \rightarrow A[-\infty, \infty]. \quad (9.2)$$

Question 1 *What is the kernels of the maps (9.2)?*

The algebra $A[-\infty, \infty]$ admits a continuous homomorphism $A[-\infty, \infty] \rightarrow A[\delta, \delta']$, where $A[\delta, \delta']$ is equipped with the discrete topology. I leave without the proof of the following pair of simple propositions.

Proposition 9.3 $A[-\infty, \infty] = \varprojlim_{\delta, \delta'} A[\delta, \delta']$.

The quotient of $A[-\infty, \infty)$ by the ideal $\mathfrak{i}([-\infty, \infty) \setminus [\delta, \infty))$ gives me the algebra $A[\delta, \infty)$. The shift (2.9) identifies $A[\delta, \infty)$ with $A[\tau(\delta), \infty)$.

Proposition 9.4

$$A[-\infty, \infty) = \varprojlim_{\delta} A[\delta, \infty) = \varprojlim_n A[(0)^{-n}, \infty).$$

The anti-involution σ of $\hat{\mathbb{E}}$ (2.11) comes from an automorphism of the linear space $S_+[z, z^{-1}]$. $\sigma(-\infty, (1)^{-n}] = [(0)^n, \infty) \supset [(1)^{n-2}, \infty)$. The action of σ on $\hat{S}_+[z, z^{-1}]$ is compatible with the relations $\Gamma^{\mathbf{s}^k}$. It implies that there is a continuous isomorphism

$$\sigma : A(-\infty, \infty] \rightarrow A[-\infty, \infty).$$

$A[(0)^0, \infty)$ is the algebra of function on the space of arcs

$$\mathcal{Arc}(0) = \left\{ \theta(z) = \sum_{\beta \in \mathbb{E}} \lambda^\beta(z) \theta_\beta = \sum_{k \geq 0} \sum_{\beta \in \mathbb{E}} \lambda^{\beta^k} \theta_\beta z^k \text{ that satisfy (2.1)} \right\}.$$

In the above formula z is treated as a formal variable and no convergence of the series is expected. The space related to $A(-\infty, (1)^0]$ is

$$\mathcal{Arc}(\infty) = \left\{ \theta(z) = \sum_{\beta \in \mathbb{E}} \lambda^\beta(z) \theta_\beta = \sum_{k \geq 0} \sum_{\beta \in \mathbb{E}} \lambda^{\beta^k} \theta_\beta z^{-k} \text{ that satisfy (2.1)} \right\}.$$

The spaces of arcs or spaces of jets are well studied in the literature. See e.g. [18]. These spaces are the inverse limits of the spaces of finite jets. In case of the cone \mathcal{C} , these spaces are

$$\mathcal{Arc}^N = \mathcal{Maps}(\text{Spec } \mathbb{C}[z]/z^{N+1}, \mathcal{C}) = \left\{ \theta(z) = \sum_{k=0}^N \sum_{\beta \in E} \lambda^{\beta^k} \theta_{\beta} z^k \text{ that satisfy (2.1) mod } z^{N+1} \right\}, N \geq 0$$

$$\mathcal{Arc}_N = \left\{ \theta(z) = \sum_{k=N}^0 \sum_{\beta \in E} \lambda^{\beta^k} \theta_{\beta} z^k \text{ that satisfy (2.1) mod } z^{N-1} \right\}, N \leq 0.$$

The projections are

$$\text{pr}^N : \mathcal{Arc}^N \rightarrow \mathcal{Arc}^{N-1}, \quad \text{pr}^N : \theta(z) \text{ mod } z^{N+1} \rightarrow \theta(z) \text{ mod } z^N.$$

As the equations (2.1) are homogeneous, the map

$$\mathbf{u}^{N-1}(\theta(z)) = z\theta(z), \quad \mathbf{u}^{N-1} : \mathcal{Arc}^{N-1} \rightarrow \mathcal{Arc}^N$$

is well defined. Similar structures exist on \mathcal{Arc}_N . The algebra of polynomial functions

$$J^N = J[(0)^0, (1)^N] = \mathcal{O}[\mathcal{Arc}^N]$$

shares its generators

$$\{\lambda^{\beta^l} | \beta \in E, 0 \leq l \leq N\}$$

with the algebra $A[(0)^0, (1)^N]$. The main difference with $A[(0)^0, (1)^N]$ is that it has as half as many relations. More precisely, the algebra J^N is the quotient of the polynomial algebra $\mathbb{C}[(0)^0, (1)^N]$ by the ideal $\mathfrak{q}[(0)^0, (1)^N]$ of relations generated by $\Gamma^{\mathbf{s}^k}$ $0 \leq l, l', k \leq N$ $\mathbf{s} \in G$ (2.2). More generally, define

$$J[\delta, (1)^N] = J[(0)^0, (1)^N] / \mathfrak{i}([(0)^0, (1)^N] \setminus [\delta, (1)^N])$$

for $[\delta, (1)^N] \subset [(0)^0, (1)^N]$.

I postulate that automorphism τ (2.9) induces an isomorphism $J[\delta, (1)^N] \cong J[\tau(\delta), (1)^{N+1}]$. This way I define $J[\delta, (1)^N]$ for the general $\delta < (1)^N$. The algebras $J_N = J[(0)^N, \delta] = \mathcal{O}[\mathcal{Arc}_N]$ are defined along the same lines. Here is the characterization of the algebra of functions on the space of formal loops.

Proposition 9.5 1. The homomorphisms $\text{pr}^{N*}, \text{pr}_N^*$ induce a direct system of algebras $\text{pr}^{N*} : J^{N-1} \rightarrow J^N$ $\text{pr}_N^* : J^{N-1} \rightarrow J^N$ such that

$$\varinjlim J^N = A[(0)^0, \infty) \quad \varinjlim J^N = A(-\infty, (1)^0]$$

and

$$\varinjlim J[\delta, (1)^N] = A[\delta, \infty) \quad \varinjlim J[(0)^N, \delta] = A(-\infty, \delta]. \quad (9.6)$$

2. The homomorphisms u_N^* induce an inverse system of algebras $u^{N-1*} : J^N \rightarrow J^{N-1}$, $u_{N-1}^* : J^N \rightarrow J^{N-1}$ such that

$$\varprojlim J^N = A[-\infty, (1)^0] \quad \varprojlim J^N = A[(0)^0, \infty].$$

- 3.

$$\varinjlim_{\substack{\text{pr}^* \rightarrow \leftarrow \\ u^*}} J^{N+N'} = A[-\infty, \infty] \quad \varinjlim_{\substack{\text{pr}^* \rightarrow \leftarrow \\ u^*}} J_{N+N'} = A(-\infty, \infty].$$

The map $J[\delta, (1)^N] \rightarrow A[\delta, \infty]$ is the part of the structure maps defining the direct limit (9.6). The composition

$$J[\delta, (1)^N] \rightarrow J[\delta, \infty] \rightarrow A[\delta, \infty] \rightarrow A[\delta, (1)^N]$$

is nothing else but the projection

$$\mathbb{C}[\delta, (1)^N]/\mathfrak{q}[\delta, (1)^N] \rightarrow \mathbb{C}[\delta, (1)^N]/\mathfrak{r}[\delta, (1)^N].$$

Question 2 Is $\text{Reg}[\delta, (1)^N]$ (4.9) regular in $J[\delta, (1)^N]$? What is the simplest maximal regular sequence in the augmentation ideal.

Note that the map σ induces an isomorphism

$$\sigma : J[\delta, (1)^N] \cong J[(0)^{-N}, \sigma(\delta)].$$

Question 3 Is $\text{Spec } J[\delta, (1)^N]$ reduced and irreducible?

The algebra $\mathbb{C}[z]/z^{N+1}$ admits a deformation $\mathbb{C}[z, q]/(z^{N+1} - q^{N+1})$. Since $\mathcal{A}rc^N$ is $\mathcal{M}aps(\text{Spec } \mathbb{C}[z]/z^{N+1}, \mathcal{C})$, the space $\mathcal{A}rc^N$ admits a "deformation" in the form

$$\mathcal{A}rc_q^N = \mathcal{M}aps(\text{Spec } \mathbb{C}[z, q]/(z^{N+1} - q^{N+1}), \mathcal{C}).$$

Note that $\text{Spec } \mathbb{C}[z, q, q^{-1}]/(z^{N+1} - q^{N+1})$ is a union of N components. This is why

$$\mathcal{M}aps(\text{Spec } \mathbb{C}[z, q, q^{-1}]/(z^{N+1} - q^{N+1}), \mathcal{C}) = \mathcal{C}^N \times \mathbb{C}^\times.$$

The algebra $J_q^N = \mathcal{O}[\mathcal{A}rc_q^N]$ is isomorphic to the quotient of $\mathbb{C}[(0)^0, (1)^N] \otimes \mathbb{C}[q]$ by the ideal of relations $\mathfrak{q}_q[(0)^0, (1)^N]$ generated by

$$\Gamma^{s^k} + q^{N+1} \Gamma^{s^{N+1+k}} \quad 0 \leq k \leq N-1 \text{ and } \Gamma^{s^N} \quad s \in G.$$

From the geometric considerations I know that the algebra $J_q^N \otimes \mathbb{C}[q, q^{-1}]$ is isomorphic to $\mathcal{O}[\mathcal{C}]^{\otimes N} \otimes \mathbb{C}[q, q^{-1}]$.

Conjecture 4 1. J_q^N is flat over $\mathbb{C}[q]$.

2. J_q^N is a Cohen-Macaulay integral domain over $\mathbb{C}[q]$.

An immediate corollary from the previous conjecture is the formula for the Hilbert series of $\mathcal{O}[\mathcal{Arc}_0^N(\mathcal{C})]$:

Corollary 9.7

1.

$$\mathcal{O}[\mathcal{Arc}_0^N(\mathcal{C})](t) = \frac{(1 + 5t + 5t^2 + t^3)^N}{(1 - t)^{11N}}.$$

2. $\mathcal{O}[\mathcal{Arc}_0^N(\mathcal{C})]$ is Gorenstein.

3. The index of the dualizing line bundle on $\text{Proj } \mathcal{O}[\mathcal{Arc}_0^N(\mathcal{C})]$ is equal to $-(11N - 3N) = -8N$.

I have only a small number of computations with *Macaulay2* to support the following conjecture.

Conjecture 5 1. More generally $J[\delta, (1)^N]$ is Cohen-Macaulay.

2. $J[\delta, (1)^N]$ is Gorenstein iff $A[\delta, (1)^N]$ is.

Remark 9.8 1. The generating series from the conjecture have been checked with *Macaulay2* for $N = 0, 1, 2, 3$.

2. In the order (8.2), defining relations of the ideal $\mathfrak{q}[\delta, (1)^N]$ do not define a quadratic Gröbner basis.

It means that most likely $J[\delta, (1)^N]$ is not Koszul.

The map of algebras

$$J[0^N, (1)^{N'}] \rightarrow A[0^N, (1)^{N'}]$$

induces a map of the minimal $P = \mathbb{C}[0^N, (1)^{N'}]$ -resolutions

$$G_\bullet[0^{-N}, \delta] \rightarrow F_\bullet[(0)^{-N}, \delta].$$

Dualizing the map and by using the conjectural Gorenstein property of J , I get an element

$$\text{th}_{A \leftarrow J} \in \text{Ext}_P^{5(N' - N)}(A, J).$$

The number $5(N' - N)$ is the codimension of $\text{Spec } A[0^N, (1)^{N'}]$ in $\text{Spec } J[0^N, (1)^{N'}]$, which conjecturally has dimension $11(N' - N + 1)$. If $N < 0 < N'$, I can use $\text{th}_{A \leftarrow J}$ to construct the map of local cohomology

$$H_{\mathfrak{a}}^i[0^N, (1)^{N'}] \rightarrow H_{\mathfrak{a}}^{i+5(N' - N)}(J[0^N, (1)^{N'}]). \quad (9.9)$$

As usual, $\mathfrak{a} = \mathfrak{i}[0^N, (1)^{-1}]$.

The next three conjectures are the central statements of this section.

- Conjecture 6** 1. The cohomology groups $H_{\mathbf{a}}^i(J[0^N, (1)^{N'}])$ for different N, N' are related by homomorphisms th as in Proposition 4.124.
2. They satisfy an analogue of the compatibility relation (4.125) that would enable me to define the limiting groups $H_{\mathbf{a}}^{i+\frac{\infty}{2}}(A_f)$.
3. The map (9.9) could be extended to

$$H_{\mathbf{a}}^{i+\frac{\infty}{2}}(A) \rightarrow H_{\mathbf{a}}^{i+\frac{\infty}{2}}(A_f).$$

Conjecture 7 In a series of works ([25],[33],[32], [26]) the authors have studied geometrically defined vertex operator algebras. These are cohomology of certain sheaves of vertex operator algebras. I conjecture that their groups defined for the smooth part of the cone \mathcal{C} coincide with $H_{\mathbf{a}}^{i+\frac{\infty}{2}}(A_f)$.

Let \mathfrak{i}_k be the ideal generated by λ^{β^s} , $s = 0, \dots, k-1$, $\beta \in E$. It defines a closed subscheme of $\mathcal{A}rc^N$ isomorphic to $\mathcal{A}rc^{N-k}$.

Let \mathfrak{j}_k be the ideal generated by λ^{β^s} , $s = k, \dots, N$, $\beta \in E$. It defines a closed subscheme of $\mathcal{A}rc^N$ isomorphic to $\mathcal{Z}(0, k-1)$ when $k < N/2$.

Conjecture 8 There is a perfect pairing

$$H_{\mathfrak{i}_k}^i(J^N) \otimes H_{\mathfrak{j}_k}^{-i+8N+16}(J^N) \rightarrow \mathbb{C}.$$

10 Final comments

In this paper, I outlined one of the approaches to local cohomology of ind scheme $\mathcal{M}aps_{poly}(\mathbb{C}^\times, \mathcal{C})$. There were other attempts to define local cohomology along a different route. Let us consider pair of Lie algebras $M = L(-\infty, \infty)$, $K = L((0)^0, \infty)$ from Section ???. One can try to define local cohomology $H_{\mathbf{a}}^{i+\frac{\infty}{2}}$ as the semi-infinite cohomology [23] of the pair $K \subset M$. Indeed, this approach demonstrated its utility in the case when \mathcal{X} is a quadric [1],[37]. The drawback of the method is the potential problem with the differential in the semi-infinite complex. Feigin's definition heavily uses the fact his Lie algebra has a finite dimension over $\mathbb{C}[z, z^{-1}]$. This condition is satisfied for the pair K, M corresponding to a quadric or any complete intersection. The space \mathcal{C} is not a complete intersection. Because of that there are serious difficulties with convergence in the formulas that define the differential in the semi-infinite complex attached to \mathcal{C} . It worthwhile to point out that the methods of this paper work for quadrics giving the answer consistent with [1],[37].

Appendix

A Algebraic and sheaf-theoretic definition of local cohomology

Review of definitions of local cohomology If R is a commutative algebra over \mathbb{C} , \mathfrak{a} an ideal of R , and N an R -module, then following [21] [29] I define the zeroeth local cohomology $H_{\mathfrak{a}}^0(N)$ module of M with supports in \mathfrak{a} to be the set of all elements of N which are annihilated by some power of \mathfrak{a} :

$$\Gamma_{\mathfrak{a}}(N) = \lim_{n \rightarrow \infty} \text{Hom}(R/\mathfrak{a}^n, N).$$

The higher local cohomology $H_{\mathfrak{a}}^i(N)$ is the i -th cohomology module of the complex obtained by applying $\Gamma_{\mathfrak{a}}(N)$ to an injective resolution of $I^{\bullet}(N)$:

$$H_{\mathfrak{a}}^i(N) := H^i(\Gamma_{\mathfrak{a}}(I(N))).$$

There is a natural isomorphism:

$$H_{\mathfrak{a}}^i(N) = \lim_{n \rightarrow \infty} \text{Ext}^i(R/\mathfrak{a}^n, N). \quad (\text{A.1})$$

According to [29] p.84 (see also [21] Th A4.1,[10] Section 20.3), if Y is an affine scheme of finite type with the algebra of functions $\mathcal{O}[Y]$ and Z is a closed subscheme with the defining ideal \mathfrak{a} , then there is a natural exact sequence

$$0 \rightarrow H_{\mathfrak{a}}^0(\mathcal{O}[Y]) \rightarrow \mathcal{O}[Y] \rightarrow H^0(Y \setminus Z, \mathcal{O}) \rightarrow H_{\mathfrak{a}}^1(\mathcal{O}[Y]) \rightarrow 0$$

and isomorphisms

$$H_{\mathfrak{a}}^i(\mathcal{O}[Y]) = H^{i-1}(Y \setminus Z, \mathcal{O}), i \geq 2.$$

H^i stands for the sheaf-theoretic cohomology.

Remark A.2 *By functoriality the symmetries of the pair $Z \subset Y$ act on $H^i(Y \setminus Z, \mathcal{O})$. In addition to this, Lie algebra \mathfrak{g} of the connected component of the group of symmetries of Y act on $H^i(Y \setminus Z, \mathcal{O})$ [30]. Note that exponents of some $l \in \mathfrak{g}$ in this case don't have to preserve Z .*

There is yet another equivalent definition of local cohomology that uses Koszul complexes. Before I give this definition I will describe two essentially isomorphic versions of the Koszul complex. They differ by multiplicative structure and gradation, which explains why I don't identify them. The first, which

will be denoted by K , is used in computation of local cohomology. The second, which will be denoted by B , is used constructing standard resolutions and computing Ext and Tor functors.

Let N be a module over the ring R . Let $\bar{x} = \{x_1, \dots, x_s\} \subset R$ be a sequence of elements. Koszul complex $K(N, \bar{x})$ is the tensor product

$$K(N, \bar{x}) = N \otimes \Lambda[\xi^1, \dots, \xi^s] = \bigoplus_{i=0}^s N \otimes \Lambda^i.$$

The differential is defined as a multiplication on the element

$$\sum_{i=1}^s x_i \xi^i.$$

The other, the homological version of the Koszul complex is

$$(B(N, \bar{x}), \partial) = (N \otimes \Lambda[\eta_1, \dots, \eta_s], \partial). \quad (\text{A.3})$$

Its differential can be concisely written by using superalgebra notations

$$\partial = \sum_{i=1}^s x_i \frac{\partial}{\partial \eta_i}.$$

It is obvious that

$$H^i(K) = H_{s-i}(B),$$

provided we made an identification of the linear spaces spanned by ξ^i and η_i .

Convention 3 Let M be a graded module over a graded algebra R . We will encounter a situation when a standard basis \bar{x} for R_1 is labeled by a subset $X \subset \hat{E}$. In this case, $B(M, X)$ and $K(M, X)$ will be abbreviations for $B(M, \{x_i | i \in X\})$, $K(M, \{x_i | i \in X\})$ respectively. To reduce clutter in notations, I will denote occasionally $B(M, \bar{x})$ and $K(M, \bar{x})$ by $B(M)$ and $K(M)$.

There is a natural map of complexes

$$K(N, \bar{x}^n) := K(N, x_1^n, \dots, x_s^n) \xrightarrow{s_n} K(N, x_1^{n+1}, \dots, x_s^{n+1}) =: K(N, \bar{x}^{n+1}). \quad (\text{A.4})$$

In degree 1 it is given by the map $f : R^s \rightarrow R^s$ multiplying the i -th component by x_i . In degree d it is exterior power of $f \wedge^d f$, which acts by multiplying a basis vector $\xi_{i_1} \wedge \dots \wedge \xi_{i_d}$ by $x_{i_1} \dots x_{i_d}$. Thus I may take the limit in each of the Koszul homology groups.

Here is the description of the precise relation of local cohomology groups and Koszul cohomology.

Proposition A.5 ([21] Appendix A4 and in [29] p.81) Suppose $\mathfrak{a} \subset R$ is an ideal generated by x_1, \dots, x_s , then

$$H_{\mathfrak{a}}^i(N) = \lim_{\substack{\longrightarrow \\ n}} H^i(K(N, \overline{x}^n)). \quad (\text{A.6})$$

Remark A.7 Formula (A.6) can be used as a constructive definition of local cohomology.

Elementary examples of computations of local cohomology

Example A.8 $K(\mathbb{C}[x], x^n)$ coincides with the two term complex $\mathbb{C}[x] \rightarrow \mathbb{C}[x], a \rightarrow x^n a$. Obviously the only nontrivial cohomology is in degree one: $H^1 = \mathbb{C}[x]/(x^n)$. The explicit form of the maps in the direct limit (A.6) is

$$\mathbb{C}[x]/(x^n) \rightarrow \mathbb{C}[x]/(x^{n+1}), \quad a \rightarrow xa.$$

The inclusion of $\mathbb{C}[x]/(x^n)$ into $x^{-1}\mathbb{C}[x^{-1}] := \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$, which is given by the formula

$$a \bmod x^n \rightarrow \frac{a}{x^n} \in \mathbb{C}[x, x^{-1}]/\mathbb{C}[x], \quad (\text{A.9})$$

identifies the later quotient with the direct limit $\lim_{\substack{\longrightarrow \\ n}} \mathbb{C}[x]/(x^n)$.

Example A.10 Let $P = \mathbb{C}[x_1, \dots, x_s]$, $N = P$ and $\mathfrak{m} = (x_1, \dots, x_s)$. By the Künneth theorem, the only nonzero local cohomology group is

$$H_{\mathfrak{m}}^s(P) \cong \bigotimes_{i=1}^s x_i^{-1} \mathbb{C}[x_i^{-1}] = x_1^{-1} \cdots x_s^{-1} \mathbb{C}[x_1^{-1}, \dots, x_s^{-1}]. \quad (\text{A.11})$$

Koszul bicomplexes One of the standard techniques of computing local cohomology with Koszul complexes relies on spectral sequences. Koszul complex can be broken down onto a bicomplex. Calculations with the spectral sequence of the bicomplex could simplify significantly computations of the cohomology of the original complex. Here are some notations used in the description of this method.

Construction A.12 Fix a decomposition of the set

$$\{x_i\} = \{x_1, \dots, x_k\} \cup \{x_{k+1}, \dots, x_s\}$$

of generators of an ideal $\mathfrak{c} \subset R$. Define bicomplexes

$$K^{i,j} = N \otimes \Lambda^i[\xi^1, \dots, \xi^k] \otimes \Lambda^j[\xi^{k+1}, \dots, \xi^s], \quad d_1(a) = \sum_{i=1}^k x_i \xi^i a, \quad d_2(a) = \sum_{i=k+1}^s x_i \xi^i a,$$

$$B_{i,j} = N \otimes \Lambda^i[\eta_1, \dots, \eta_k] \otimes \Lambda^j[\eta_{k+1}, \dots, \eta_s], \quad \partial_1(a) = \sum_{i=1}^k x_i \frac{\partial a}{\partial \eta_i}, \quad \partial_2(a) = \sum_{i=k+1}^s x_i \frac{\partial a}{\partial \eta_i}.$$

Their diagonal complexes coincide with K^\bullet and B_\bullet respectively.

Some free $\mathbb{C}[x^1, \dots, x^n]$ -modules In this paragraph will construct some free $\mathbb{C}[x^1, \dots, x^n]$ -modules. A free $\mathbb{C}[x^1, \dots, x^n]$ -modules N has a property that the sequence $\{x^1, \dots, x^n\}$ is N -regular, which makes verification of regularity a trivial task. We will use this observation in the main text.

A polynomial algebra $H = \mathbb{C}[x^1, \dots, x^n]$ is a Hopf algebra. The diagonal $\Delta : H \rightarrow H \otimes H$ and the antipode $\iota : H \rightarrow H$ are

$$\Delta(x_i) = x^i \otimes 1 + 1 \otimes x^i, \iota(x^i) = -x^i.$$

If M, N are H -modules, then $M \otimes N$ is also an H -module with respect to three module structures: $T_0(a)(m \otimes n) := \sum a'_k m \otimes a''_k n$, $\Delta(a) = \sum a'_k \otimes a''_k$ or $T_1(a)(m \otimes n) := \sum \iota(a'_k) m \otimes a''_k n$ or $T_2(a)(m \otimes n) := \sum a'_k m \otimes \iota(a''_k) n$.

Lemma A.13 *Fix two H -modules M and N .*

1. *If M is free, then $M \otimes N$ is also free with respect to all the three structures.*
2. *$B(N \otimes H, (\underline{x}^i))$ is a free H resolution of N .*

$$\begin{aligned} \underline{x}^i &= x^i \otimes 1 - 1 \otimes x^i, \\ \underline{\underline{x}}^i &= x^i \otimes 1 + 1 \otimes x^i. \end{aligned} \tag{A.14}$$

3. *Suppose $H_1 = \mathbb{C}[x^1, \dots, x^n]$, $H_2 = \mathbb{C}[y_1, \dots, y_{n'}]$ and $H = H_1 \otimes H_2$. If graded H -modules M, N have the property that $M = H_1 \otimes M_2$, M_2 is an H_2 -module and $N|_{H_2}$ is free. Then $M \otimes N$ is free over H . In particular, $(\underline{x}_i, \underline{y}_j)$ is regular.*

Proof.

1. I prove for the structure T_0 only. It suffice to prove the statement when $M = H$. The linear space $L = H \otimes N$ has two H -module structures. The diagonal structure L_Δ was described above. The other structure L_{free} is $a(h \otimes n) := (ah) \otimes n$. L_{free} is a free module. The isomorphism $f_H : L_\Delta \rightarrow L_{free}$ between L_Δ and L_{free} maps $h \otimes n$ to $\sum h_k \otimes \iota(h'_k) n$. The inverse is $f_H^{-1}(h \otimes n) = \sum h_k \otimes h'_k n$.
2. By the first item $N \otimes H$ is H -free and (\underline{x}^i) is regular. Thus $B(N \otimes H, (\underline{x}^i))$ has no higher cohomology and the zero cohomology is equal to N .
3. By the second item $B(N \otimes M, (\underline{x}^i, \underline{y}^j))$ computes $\text{Tor}_i^H(M, N)$. Note that H is flat over H_2 and $M = H \otimes_{H_2} M_1$. Thus

$$\text{Tor}_i^H(M, N) = \text{Tor}_i^H(H \otimes_{H_2} M_2, N) \cong \text{Tor}_i^{H_2}(M_2, N).$$

N is H_2 -free. $\text{Tor}_i^{H_2}(M_2, N) = 0, i \geq 1$. This implies that $N \otimes M$ is H -free and $(\underline{x}^i, \underline{y}^j)$ is regular.

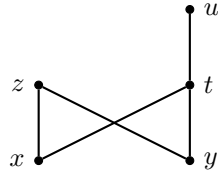
■

B Generalities about Hibi rings

For more details on Hibi rings the reader can consult the survey [22] or the original article [27].

Let P be a poset. Element y covers x if $y > x$ and there is no $z \in P$ with $y > z > x$. In this case we write $y \succ x$.

The definition of Hasse diagram uses only $y \succ x$ relation (see [7] p.5 for details). Here is an example of the Hasse diagram of a poset P with 5 elements, x, y, z, t, u with $z \succ x, z \succ y, t \succ y, t \succ x, u \succ t$.



Definition B.1 A graded poset is a partially ordered set (poset) P equipped with a rank function ρ from P to $\mathbb{Z}_{\geq 0}$ satisfying the following two properties:

1. The rank function is compatible with the ordering, meaning that for every x and y in the order with $x < y$, it must be the case that $\rho(x) < \rho(y)$, and
2. If $x < y$, then $\rho(y) = \rho(x) + 1$.

Example B.2 \hat{E} is a graded poset:

the function $\rho : \hat{E} \rightarrow \mathbb{Z}$ in this case is defined by the rule

$$\begin{aligned}
 \rho((0)^r) &= \rho((3)^{r-1}) = 8r \\
 \rho((12)^r) &= \rho((2)^{r-1}) = 8r + 1 \\
 \rho((13)^r) &= \rho((1)^{r-1}) = 8r + 2 \\
 \rho((14)^r) &= \rho((23)^r) = 8r + 3 \\
 \rho((15)^r) &= \rho((24)^r) = 8r + 4 \\
 \rho((25)^r) &= \rho((34)^r) = 8r + 5 \\
 \rho((35)^r) &= \rho((5)^r) = 8r + 6 \\
 \rho((45)^r) &= \rho((4)^r) = 8r + 7.
 \end{aligned} \tag{B.3}$$

Example B.4 $B[-\infty, \infty]$ (see page 24) is a graded poset:

the function $\rho_{B[-\infty, \infty]} : B[-\infty, \infty] \rightarrow \mathbb{Z}$ is

$$\rho((14)^r) = \rho((1)^{r-1}) = 4r + 1$$

$$\rho((0)^r) = \rho((2)^{r-1}) = 4r$$

$$\rho((45)^{r-1}) = \rho((5)^{r-1}) = 4r - 1$$

$$\rho((15)^{r-1}) = \rho((34)^{r-1}) = 4r - 2.$$

Let

$$C : x' = x_1 < \cdots < x_t = x$$

be a chain in P (i.e. a totally ordered subset of P). I say that C is a chain descending from x' . The length of C will be

$$|C| - 1. \tag{B.5}$$

The rank of a poset P , denoted by $\text{rk}(P)$, is the supremum of the lengths of all chains contained in P . A poset is called pure if all maximal chains have the same length. The *height* of an element $\alpha \in P$, denoted $\rho_P(\alpha)$ is:

$$\rho_P(\alpha) = \sup\{\text{length of chains descending from } \alpha\}.$$

Let P be a poset and $x, y \in P$. An *upper bound* of x, y is an element $z \in P$ such that $z \geq x$ and $z \geq y$. If the set $\{z \in P : z \text{ is an upper bound of } x \text{ and } y\}$ has the least element, is called the *join* of x and y , and it is denoted $x \vee y$. By duality, one defines the *meet* $x \wedge y$ of two elements x, y in a poset.

Definition B.6 Let L be a lattice. L is called *distributive* if satisfies one of the equivalent conditions:

$$(i) \text{ for any } x, y, z \in L, x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$$

$$(ii) \text{ for any } x, y, z \in L, x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A subset α of a poset P is called an *order ideal* or *poset ideal* if it satisfies the following condition: for any $x \in \alpha$ and $y \in P$, if $y \leq x$, then $y \in \alpha$. The set of all order ideals of P is denoted $\mathcal{I}(P)$. The union and intersection of two order ideals are obviously order ideals. Therefore, $\mathcal{I}(P)$ is a distributive lattice with the union and intersection.

Given a lattice L , an element $x \in L$ is called *join-irreducible* if $x \neq \min L$ and whenever $x = y \vee z$ for some $y, z \in L$, we have either $x = y$ or $x = z$.

Theorem B.7 (Birkhoff) *Let L be a distributive lattice and $P = B(L)$ its subposet of join-irreducible elements. Then L is isomorphic to $\mathcal{I}(P)$.*

Let L be a lattice. The generators λ^α of the algebra $Hibi(L)$ are labeled by the vertices of the lattice L and the straightening laws are the so called Hibi relations:

$$\lambda^\alpha \lambda^\beta = \lambda^{\alpha \wedge \beta} \lambda^{\alpha \vee \beta}, \forall \alpha, \beta \in L \mid \alpha \not\leq \beta \text{ and } \alpha \not\geq \beta. \quad (\text{B.8})$$

Remark B.9 Fix a distributed lattice $L = \mathcal{I}(P)$.

1. ([27]) Hibi rings are toric algebras with straightening laws, thus normal Cohen-Macaulay domains.
2. Krull dimension $\dim(Hibi(L))$ is equal to $\text{rk} L + 1$.
3. ([27]) $Hibi(L) = Hibi(\mathcal{I}(P))$ is Gorenstein if and only if P is pure.

C General constructions with D-modules

Weyl algebras Recall that the Weyl algebra $W(U, \omega)$ on a finite-dimensional symplectic linear space (U, ω) is the quotient

$$\bigoplus_{i \geq 0} U^{\otimes i} / ([u, u'] - \omega(u, u')).$$

When an orthogonal sum decomposition of the symplectic space $U = U_1 \oplus U_2$ is given, there is a canonical isomorphism of algebras $W(U) \cong W(U_1) \otimes W(U_2)$.

Symplectization of a finite-dimensional vector space V is a direct sum $U(V) = V + V^*$. It carries a symplectic form $\omega(v + v^*, v' + v'^*) = \langle v^*, v' \rangle - \langle v, v'^* \rangle$. $D(V) := W(U(V))$ is the algebra of (polynomial) differential operators. The direct sum decomposition $V = V_1 + V_2$ leads to orthogonal decomposition of the $U(V)$ and an isomorphism $D(V) \cong D(V_1) \otimes D(V_2)$. In particular, $D(V_1)$ becomes a subalgebra of $D(V)$. Let V be a linear space equipped with a nondegenerate symmetric inner product (\cdot, \cdot) . If $V' \subset V$ is not degenerate, i.e. contains no isotropic subspaces, then $V = V' + V'^\perp$ and $D(V')$ is a subalgebra in $D(V)$.

The algebra $D(V)$ is isomorphic as a linear space to the tensor product of two symmetric algebras

$$\text{Sym}[V] \otimes \text{Sym}[V^*] \cong \text{Sym}[V^*] \otimes \text{Sym}[V]. \quad (\text{C.1})$$

I interpret the factor $\text{Sym}[V^*] = \mathbb{C}[V] = \mathcal{O}[V]$ as the space of polynomial function on V . The factor $\text{Sym}[V] = \mathbb{C}[V^*]$ corresponds to differential operators with constant coefficients.

Modules over the Weyl algebra There are two standard $D(V)$ -modules. The first is the tautological representations of $D(V)$ in the polynomial algebra $\mathbb{C}[V]$. The unit element $\varpi_V = 1 \in \mathbb{C}[V]$ is a cyclic generator that satisfies

$$v\varpi_V = 0, \quad v \in V.$$

Because of the tensor decomposition (C.1), these are defining relations for $\mathbb{C}[V]$.

The module which I denote by $\mathbb{C}[V]^{-1}$ has a cyclic generator ϖ_{V^*} (it is usually called Dirac δ -function) that satisfies

$$v^*\varpi_{V^*} = 0, v^* \in V^*. \quad (\text{C.2})$$

$\mathbb{C}[V]^{-1}$ can be interpreted as a space of algebraic distributions on V with support at 0. This opens a way for constructing numerous examples of $D(V)$ -modules.

Fix a pair of embedding $\mathbf{a} : V \rightarrow V' \cong \text{Im } \mathbf{a} \oplus \text{Im } \mathbf{a}^\perp$, $\mathbf{b} : U \rightarrow U' \cong \text{Im } \mathbf{b} \oplus \text{Im } \mathbf{b}^\perp$. In the presence of dot product, they define inclusions

$$\mathbf{a} : \mathbb{C}[V] \rightarrow \mathbb{C}[V'], \quad x \rightarrow \mathbf{a}(x) \otimes 1, \quad (\text{C.3})$$

$$\mathbf{b} : \mathbb{C}[U]^{-1} \rightarrow \mathbb{C}[U']^{-1}, \quad y \rightarrow \mathbf{b}(y) \otimes \varpi_{\text{Im } \mathbf{b}^\perp} \text{ and}$$

$$\mathbf{a} \otimes \mathbf{b} : \mathbb{C}[V] \otimes \mathbb{C}[U]^{-1} \rightarrow \mathbb{C}[V'] \otimes \mathbb{C}[U']^{-1}. \quad (\text{C.4})$$

A model for $\mathbb{C}[V]^{-1}$ Suppose that V is a one-dimensional space with a coordinate x .

$$0 \rightarrow \mathbb{C}[x] \rightarrow \mathbb{C}[x^{\pm 1}] \rightarrow \mathbb{C}[x^{\pm 1}]/\mathbb{C}[x] \rightarrow 0$$

is a short exact sequence of $D[V]$ -modules. I denote $\mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$ by $x^{-1}\mathbb{C}[x^{-1}]$. It is easy to see that $x^{-1}\mathbb{C}[x^{-1}]$ is cyclicly generated over $D[V]$ by $\frac{1}{x}$ and as linear spaces

$$x^{-1}\mathbb{C}[x^{-1}] \cong \mathbb{C}[V]^{-1}.$$

More generally, if V is spanned by the set $A = \{x_1, \dots, x_n\}$, then the module

$$\mathbb{C}[A^{-1}] := \bigotimes_{x_i \in A} x_i^{-1} \mathbb{C}[x_i^{-1}] \cong \mathbb{C}[V]^{-1} \quad (\text{C.5})$$

is cyclicly generated over $D[V]$ by $\varpi_V = \prod_{x_i \in A} x_i^{-1}$. The module $\mathbb{C}[A^{-1}]$ has a \mathbb{C} basis $\{x_1^{-1-k_1} \dots x_n^{-1-k_n} | k_i \geq 0\}$. There is a product

$$\begin{aligned}
& \mathbb{C}[A] \otimes \mathbb{C}[A^{-1}] \rightarrow \mathbb{C}[A^{-1}] \text{ and more generally} \\
& (\mathbb{C}[A \cup B] \otimes \mathbb{C}[C^{-1}]) \otimes (\mathbb{C}[A \cup C] \otimes \mathbb{C}[B^{-1}]) \cong \\
& \cong \mathbb{C}[A] \otimes \mathbb{C}[A] \otimes \mathbb{C}[B] \otimes \mathbb{C}[B^{-1}] \otimes \mathbb{C}[C] \otimes \mathbb{C}[C^{-1}] \rightarrow \\
& \rightarrow \mathbb{C}[A] \otimes \mathbb{C}[B^{-1}] \otimes \mathbb{C}[C^{-1}] \cong \mathbb{C}[A] \otimes \mathbb{C}[(B \cup C)^{-1}].
\end{aligned} \tag{C.6}$$

The Lie algebra of differential operators $D[V]$ contains a copy of \mathfrak{gl}_n generated by $x_i \partial_j$. It acts on the generator $x_1^{-1} \cdots x_n^{-1}$ of $\mathbb{C}[V]^{-1}$ by

$$\begin{aligned}
ax_1^{-1} \cdots x_n^{-1} &= -\text{tr}(a)x_1^{-1} \cdots x_n^{-1} \\
a &= \sum_{i,j=1}^n a_{ij}^i x_i \partial_j, \text{tr}(a) = \sum_{i=1}^n a_{ii}^i.
\end{aligned} \tag{C.7}$$

D Pairing between limits

Pairing between direct and inverse systems Let A be a partially ordered set. I will be interested in a pairing between the direct system of complex linear spaces $\iota^{\alpha',\alpha} : F^\alpha \rightarrow F^{\alpha'}, \alpha \leq \alpha'$ and the inverse system $\pi_{\alpha,\alpha'} : G_{\alpha'} \rightarrow G_\alpha, \alpha \leq \alpha'$ (see [16], [49] and [8] for terminology and definitions). I set

$$F := \varinjlim F^\alpha, \quad G := \varprojlim G_\alpha.$$

I assume that the pairing

$$F^\alpha \otimes G_\alpha \xrightarrow{(\cdot, \cdot)} \mathbb{C} \tag{D.1}$$

is compatible with π and ι :

$$(\iota^{\alpha',\alpha}(f^\alpha), g_{\alpha'}) = (f^\alpha, \pi_{\alpha,\alpha'} g_{\alpha'}). \tag{D.2}$$

Lemma D.3 *The pairing (D.1) between the direct system $\{F^\alpha\}$ and the inverse system $\{G_\alpha\}$ induces a pairing between limits*

$$F \otimes G \rightarrow \mathbb{C}.$$

Proof. By definition of the inverse limit $g \in G$ is represented by a sequence $g_\alpha \in G_\alpha$ which satisfies coherence condition $\pi_{\alpha,\alpha'} g_{\alpha'} = g_\alpha$. The direct limit (see e.g. [16] p.11 for details) is equipped with maps $\psi^\alpha : F^\alpha \rightarrow F$. For any $f \in F$ there is α and f^α such that $\psi^\alpha f^\alpha = f$. I define

$$(f, g) := (f^\alpha, g_\alpha).$$

By universality of the direct limit

$$f = \psi^\alpha f^\alpha = \psi^{\alpha'} f^{\alpha'}, \alpha \leq \alpha' \Rightarrow f^{\alpha'} = \iota^{\alpha' \alpha} f^\alpha$$

This together with (D.2) verifies correctness of the definition. ■

Proposition D.4 *Let us suppose that in addition to assumptions of Lemma D.3 the pairings (D.1) are nondegenerate and G_α satisfies Mittag-Leffler condition and the set A is totally ordered. Then the pairing (D.3) has trivial right and left kernels.*

Proof. Fix $f \in F$ such that $(f, g) = 0$ for all $g \in G$. Choose $f^\alpha \in F^\alpha, \psi^\alpha = f$. I use f^α to define a map $f : G_{\alpha'} \rightarrow \mathbb{C}_{\alpha'}, \alpha \leq \alpha'$, where $\mathbb{C}_\alpha = \mathbb{C}$ is a constant inverse system. The map is defined by the formula

$$g_{\alpha'} \rightarrow (f^\alpha, \pi_{\alpha, \alpha'} g_{\alpha'})$$

As A is totally ordered

$$\varprojlim_{A \geq \alpha} G_\beta = \varprojlim_A G_\beta$$

The short exact sequence of inverse systems $\{0\} \rightarrow \text{Ker}_\alpha \rightarrow G_\alpha \rightarrow \mathbb{C}_\alpha \rightarrow \{0\}$ induces a long exact sequence of limits:

$$\{0\} \rightarrow \varprojlim \text{Ker}_\alpha \rightarrow \varprojlim G_\alpha \xrightarrow{f} \mathbb{C} \rightarrow \varprojlim^1 \text{Ker}_\alpha \rightarrow \dots \quad (\text{D.5})$$

Fix $\beta \geq \alpha$. Let $\beta' \geq \beta$ be the index such that $\text{Im} \pi_{\beta, \gamma} = \text{Im} \pi_{\beta, \gamma'}$ for $\gamma, \gamma' \geq \beta'$. Mittag-Leffler condition postulates existence of such β' . Pick $g_\gamma \in \pi_{\beta, \gamma}^{-1}(\text{Im} \pi_{\beta, \gamma} \cap \text{Ker}_\beta)$. Equality (D.2) implies that $g_\gamma \in \text{Ker}_\gamma$. I conclude that $\{\text{Ker}_\gamma\}$ satisfies Mittag-Leffler condition. It implies (see e.g. [49] Proposition 3.5.7) that $\varprojlim^1 \text{Ker}_\alpha = \{0\}$ and the map f (D.5) is nonzero. ■

D.1 Pairing between bidirect systems

I consider a poset A and a double indexed family of linear spaces G_α^β with $\alpha, \beta \in A$. I suppose that for every $\beta_0 \in A$ $G_{\alpha}^{\beta_0}$ is an inverse system. Symmetrically $G_{\alpha_0}^\beta$ is a direct system for every $\alpha_0 \in A$. Maps ι and π satisfy compatibility relation which says that for every $\alpha' \leq \alpha$ and $\beta \leq \beta'$ the square

$$\begin{array}{ccc} G_\alpha^\beta & \xrightarrow{\pi_{\alpha', \alpha}^\beta} & G_{\alpha'}^\beta \\ \downarrow \iota_{\alpha}^{\beta, \beta'} & & \downarrow \iota_{\alpha'}^{\beta', \beta} \\ G_\alpha^{\beta'} & \xrightarrow{\pi_{\alpha', \alpha}^{\beta'}} & G_{\alpha'}^{\beta'} \end{array} \quad (\text{D.6})$$

is commutative. Such double indexed family will be called a *bidirect system*. I will use the following abbreviations:

$$G^\beta = \varprojlim_\alpha G_\alpha^\beta, \quad G_\alpha = \varinjlim_\beta G_\alpha^\beta$$

G^β has a structure of a direct system and G_α of an inverse system. I will refer to G^β (G_α) as a contraction of G_α^β over lower(upper) index. Occasionally it will be convenient to think about G^β as of bidirect system whose π -maps are identities. Likewise G_α is a bidirect system whose ι maps are identities.

A pairing between two bidirect system (F, π, ι) , (G, π, ι) is a linear map

$$F_\alpha^\beta \otimes G_\beta^\alpha \rightarrow \mathbb{C} \quad (\text{D.7})$$

which satisfies

$$\begin{aligned} (\pi_{\alpha', \alpha}^\beta f_\alpha^\beta, g_\beta^{\alpha'}) &= (f_\alpha^\beta, \iota_{\beta, \alpha'}^{\alpha, \alpha'} g_\beta^{\alpha'}), \quad \alpha' \leq \alpha \\ (\iota_{\alpha, \beta'}^{\beta', \beta} f_\alpha^\beta, g_\beta^{\alpha'}) &= (f_\alpha^\beta, \pi_{\beta', \beta}^{\alpha', \beta} g_\beta^{\alpha'}), \quad \beta \leq \beta' \end{aligned} \quad (\text{D.8})$$

Construction from Lemma D.3 defines pairings

$$F_\alpha \otimes G^\alpha \rightarrow \mathbb{C}, \quad F^\alpha \otimes G_\alpha \rightarrow \mathbb{C}$$

between pairs of systems $(F_\alpha, \pi), (G^\alpha, \iota)$ and $(F^\alpha, \iota), (G_\alpha, \pi)$. Applying Lemma D.3 one more time I get pairings

$$\varprojlim \varinjlim F \otimes \varinjlim \varprojlim G \xrightarrow{(\cdot, \cdot)^L} \mathbb{C}, \quad \varinjlim \varprojlim F \otimes \varprojlim \varinjlim G \xrightarrow{(\cdot, \cdot)^R} \mathbb{C} \quad (\text{D.9})$$

There is a four-term exact sequence

$$\{0\} \rightarrow \text{Ker}_\beta^\alpha \rightarrow G_\beta^\alpha \xrightarrow{\kappa} G_\beta \rightarrow \text{coKer}_\beta^\alpha \rightarrow \{0\} \quad (\text{D.10})$$

Applying \varprojlim to κ I get a map $\kappa : G^\alpha \rightarrow \varprojlim \varinjlim G$. If now I use \varinjlim I get a map

$$\kappa : \varinjlim \varprojlim G \rightarrow \varprojlim \varinjlim G$$

Remark D.11 All the above considerations go through if we reduce the range of the bi-index in G_β^α to a subset $\{(\alpha, \beta) | \alpha \geq \beta\} \subset A \times A$.

Example D.12 A bidirect system of finite-dimensional linear spaces is called generic if the ranks of all the structure maps are maximal. Suppose $A = \mathbb{Z}$.

1. Let

$$H_\beta^\alpha = \begin{cases} \mathbb{C} & \beta \geq \alpha \\ \{0\} & \beta < \alpha \end{cases}$$

be a generic bidirect system. In this case $\varprojlim H \cong \{0\}$, $\varinjlim H \cong \mathbb{C}$.

2. Let

$$F_\beta^\alpha = \begin{cases} \{0\} & \beta \geq \alpha \\ \mathbb{C} & \beta < \alpha \end{cases}$$

be a generic bidirect system. In this case $\varprojlim H \cong \mathbb{C}$, $\varinjlim F \cong \{0\}$.

3. Let

$$G_\beta^\alpha = \begin{cases} \mathbb{C} & \beta \neq \alpha \\ \{0\} & \beta = \alpha \end{cases}$$

be a generic bidirect system. In this case $G_\beta \cong G^\alpha \cong \varprojlim G \cong \varinjlim G \cong \mathbb{C}$ but $\kappa = 0$.

In my application I am mostly interested in systems such that

$$\dim_{\mathbb{C}} G_\beta^\alpha \leq C \tag{D.13}$$

for some constant C .

Proposition D.14 Suppose that the bidirect system G_β^α satisfies (D.13). Then there is a short exact sequence

$$\{0\} \rightarrow \varinjlim \text{Ker}_\beta^\alpha \rightarrow \varinjlim G \xrightarrow{\kappa} \varprojlim G \rightarrow \varprojlim \text{coKer}_\beta^\alpha \rightarrow \{0\} \tag{D.15}$$

Proof. Fix α in (D.10). Mittag-Leffler condition for all inverse systems in (D.10) follows trivially from (D.13). If we use now \varprojlim we will get an exact four term sequence

$$\{0\} \rightarrow \text{Ker}^\alpha \rightarrow G^\alpha \xrightarrow{\kappa} \varprojlim G \rightarrow \text{coKer}^\alpha \rightarrow \{0\}$$

\varinjlim is an exact functor in our context ([49] Theorem 2.6.15). This verifies (D.15) ■

Proposition D.16 Let $G_\beta^\alpha, \beta, \alpha \in \mathbb{Z}$ is a bidirect system.

1. If $\iota_\beta^{\alpha', \alpha}$ are injective, then the map κ is also injective.
2. If $\iota_\beta^{\alpha', \alpha} \pi_{\beta, \beta'}^\alpha$ are bijections, then the map κ is an isomorphism.

Proof. Let g_β^α be a coherent system of element $\alpha \geq \alpha_0$ that represents $g \in \varinjlim G$. I extend it to a system that is defined for all β, α such that $\alpha \geq \alpha_0$. As $\iota_\beta^{\alpha', \alpha}$ are embeddings the system is determined by a subsystem defined for $\{\alpha, \beta | \alpha \geq A(\beta)\}$. The function $A : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfies $A(\beta') > A(\beta)$ for $\beta' > \beta$. Such subsystem coincides with κg . In particular if such a subsystem is zero then $g = 0$.

The second statement is a special case [24] Theorem 5.6. ■

Proposition D.17 Suppose $A = \mathbb{Z}$. Pairing $(\kappa a, b)_L$ and $(a, \kappa b)_R$ coincide. They define a bilinear map

$$\varinjlim F \otimes \varinjlim G \rightarrow \mathbb{C}$$

Proof. The element $f \in \varinjlim F$ and $f' \in \varinjlim F$ is represented by coherent collections $\{f_\beta^\alpha\}$ and $\{f'_\beta^\alpha\}$ such that the range of indices $\{(\alpha, \beta)\}$ belong to subsets $\text{supp} f, \text{supp} f' \subset \mathbb{Z} \times \mathbb{Z}$.

$$\text{supp} f = \{\alpha, \beta | \beta \geq B_f(\alpha), \alpha \geq \alpha_0\}, \quad B_f(\alpha') \geq B_f(\alpha) \text{ if } \alpha' \geq \alpha$$

$$\text{supp} f' = \{\alpha, \beta | \alpha \geq A_{f'}(\beta), \beta \geq \beta_0\}, \quad A_{f'}(\beta + 1) \geq A_{f'}(\beta) \text{ if } \beta' \geq \beta$$

$A : \mathbb{Z} \rightarrow \mathbb{Z}, B : \mathbb{Z} \rightarrow \mathbb{Z}$ are some functions. One can use structure maps of the bidirect system to unambiguously extend support of the collection. In particular support of f can be extended to $\{\alpha, \beta | \alpha \geq A(\beta) = \alpha_0\}$. The procedure of extension coincides with the map κ . The pairing $(\kappa f, g)$ ($f, \kappa g$) are computed by performing the extension procedure on $\{f_\beta^\alpha\}$ or $\{g_\beta^\alpha\}$ respectively followed by coupling $(f_\beta^\alpha, g_\alpha^\beta)$. Equations (D.8) imply that $(f_\beta^\alpha, g_\alpha^\beta) = (f_{\beta'}^{\alpha'}, g_{\alpha'}^{\beta'})$ for $\alpha > \beta$ and $\alpha' < \beta'$. This is the assertion we are trying to prove. ■

Proposition D.18 Suppose that the bidirect system $F_\alpha^\beta, G_\beta^\alpha$ satisfies (D.13), the pairing (D.7) is not degenerate, and A is totally ordered. Then the pairing (D.9) are not degenerate.

Proof. I contract $F_\alpha^\beta, G_\beta^\alpha$ over index β . The result is a pair of systems with a pairing. It follows from finite-dimensionality (D.13) that inverse systems G_β^α satisfy Mittag-Leffler condition. By Proposition D.4 the induced pairing $F_\alpha \otimes G^\alpha \rightarrow \mathbb{C}$ is not degenerate. It follows from (D.13) that $\dim_{\mathbb{C}} G_\alpha, \dim_{\mathbb{C}} F_\alpha \leq C$. I repeat the previous arguments with contraction one more time but now with the index α . This way I get nondegeneracy of the first pairing (D.9).

Nondegeneracy of the other pairing in (D.9) is proved similarly. ■

E Some technical lemmas

Definition E.1 For $\alpha \in \hat{E}$ define

$$\text{CL}^\pm(\alpha) := \{\beta \in \hat{E} | (\alpha, \beta) \text{ is a clutter and } \pm \rho(\beta) \leq \pm \rho(\alpha)\}.$$

Definition E.2 Define the subsets of \hat{E}

$$M_1^- := \left\{ \alpha \in \hat{E} \mid |\text{CL}^-(\alpha)| = 1 \right\} = \{(13)^r, (14)^r, (24)^r, (34)^r, (35)^r, (45)^r, (3)^r, (2)^r \mid r \in \mathbb{Z}\},$$

$$M_2^- := \left\{ \alpha \in \hat{E} \mid |\text{CL}^-(\alpha)| = 2 \right\} = \{(4)^r, (12)^r, (23)^r, (25)^r\},$$

$$M_3^- := \left\{ \alpha \in \hat{E} \mid |\text{CL}^-(\alpha)| = 3 \right\} = \{(0)^r, (1)^r, (15)^r, (5)^r\},$$

$$M_1^+ := \left\{ \alpha \in \hat{E} \mid |\text{CL}^+(\alpha)| = 1 \right\} = \{(12)^r, (13)^r, (23)^r, (24)^r, (25)^r, (35)^r, (4)^r, (3)^r \mid r \in \mathbb{Z}\},$$

$$M_2^+ := \left\{ \alpha \in \hat{E} \mid |\text{CL}^+(\alpha)| = 2 \right\} = \{(14)^r, (2)^r, (45)^r, (34)^r\},$$

$$M_3^+ := \left\{ \alpha \in \hat{E} \mid |\text{CL}^+(\alpha)| = 3 \right\} = \{(0)^r, (1)^r, (15)^r, (5)^r\}.$$

Lemma E.3 1. $1 \leq |\text{CL}^\pm(\alpha)| \leq 3, \forall \alpha \in \hat{E}$. The sets $\text{CL}^\pm(\alpha)$ are totally ordered.

2. Let $\alpha := \gamma \wedge \delta'$. Then α doesn't depend on $\gamma \in \text{CL}^-(\delta')$. In addition, α satisfies $\alpha \leq \delta'$.

3. $\rho(\gamma) \leq \rho(\text{CL}^-(\gamma))$, $\rho(\gamma) \geq \rho(\text{CL}^+(\gamma))$. Equality is achieved on some element of $\text{CL}^\pm(\gamma)$.

4.

$$\hat{E} = M_1^- \sqcup M_2^- \sqcup M_3^- = M_1^+ \sqcup M_2^+ \sqcup M_3^+,$$

$$M := M_1^\pm \cap M_1^\mp = \{(3)^r, (13)^r, (24)^r, (35)^r\} \neq \emptyset,$$

$$M \sqcup M_2^\pm = M_1^\mp,$$

$$M_2^- \cap M_2^+ = \emptyset,$$

$$M_3^+ = M_3^-.$$

5.

$$\forall \alpha \in M_2^- \cup M_3^- \quad \exists! \beta \in M_1^+ \text{ such that } \alpha \leq \beta,$$

$$\forall \alpha \in M_2^+ \cup M_3^+ \quad \exists! \beta \in M_1^- \text{ such that } \alpha \geq \beta.$$

6. If $\rho(\gamma) = \rho(\gamma')$, $\gamma \neq \gamma'$, then one of the element belongs to M_1^- and the other to M_2^- or one of the element belongs to M_1^+ and the other to M_3^+ .

7. If $\alpha, \beta \in M_1^+$ and $\rho(\alpha) \equiv \rho(\beta) \pmod{2}$, then $|\text{CL}^-(\alpha)| = |\text{CL}^-(\beta)|$.

Proof. The proof is a straightforward verification with the diagram (2.5). ■

The next lemma is very technical but straightforward.

Lemma E.4 Fix $\delta < \beta$ such that $\delta \in M = M_1^- \cap M_1^+$.

1. There is a unique element $\delta' \in M_1^+$ such that $\delta \leq \delta'$.
2. $|\text{CL}^-(\delta')| = 2$.
3. $\lambda^{\delta'}$ is nonzero in $A(\delta, \beta]$.
4. There are $\gamma < \gamma'$ such that $\lambda^\gamma, \lambda^{\gamma'}$ are nonzero in $A(\delta, \beta]$ and

$$\lambda^\gamma \lambda^{\delta'} = \lambda^\gamma \lambda^{\delta'} = 0.$$

5. Elements $\lambda^\gamma, \lambda^{\gamma'}$ belong to the kernel of the projection $A(\delta, \beta] \rightarrow A[\delta', \beta]$ which corresponds to inclusion $[\delta', \beta] \subset (\delta, \beta]$.
6. $(\delta, \beta] = \text{CL}^-(\delta') \sqcup [\delta', \beta]$.
7. $(\delta, \beta] = \text{CL}^-(\gamma) \sqcup [\gamma, \beta]$, $\text{CL}^-(\gamma) = \{\delta'\}$.

Proof.

1. $\delta' \in M_1^+ = M \sqcup M_2^-$ (item (4) Lemma E.3) exists because $(M_1^+)_{\delta <}$ is a well ordered set (see (2.40)).
2. The set of all such δ' (see the diagram (2.5)) is $\{(25)^r, (4)^r, (12)^r, (23)^r\} = M_2^-$.
3. $\lambda^{\delta'}$ is one of the standard monomials in $A(\delta, \beta]$ (see Proposition 2.18).
4. I choose $\gamma, \gamma' \in \text{CL}^-(\delta')$. By the item (1) of Lemma E.3, $\text{CL}^-(\delta')$ is totally ordered. I can assume $\gamma < \gamma'$. By item (2) of the same lemma, $\delta' \wedge \gamma = \delta' \wedge \gamma' = \alpha, \delta \leq \alpha \leq \delta'$. Since $\delta \leq \delta'$

$$\delta' \wedge \gamma = \delta' \wedge \gamma' = \delta. \tag{E.5}$$

Relation (2.15) in $A[\delta, \beta]$ associated with the clutters (δ', γ) (δ', γ') contains no terms under summation sign because

$$\{\alpha \in [\delta, \beta] | \alpha < \delta' \wedge \gamma = \delta\} = \{\alpha \in [\delta, \beta] | \alpha < \delta' \wedge \gamma' = \delta\} = \emptyset.$$

In $A(\delta, \beta]$ (2.17)

$$\lambda^\gamma \lambda^{\delta'} = \pm \lambda^\delta \lambda^{\delta' \vee \gamma} = 0.$$

By the same reasons, $\lambda^{\gamma'} \lambda^{\delta'} = 0$.

5. $A[\delta', \beta]$ has no zero divisors [36].
6. From (E.5), I conclude that $\text{CL}^-(\delta')$ and $[\delta', \beta]$ are subsets of $(\delta, \beta]$. If $\alpha \in (\delta, \beta] \setminus [\delta', \beta]$, then, by definition of CL , $\alpha \in \text{CL}^-(\delta')$.
7. By construction $\delta < \gamma$. By item (3) Lemma E.3, $\rho(\delta') = \rho(\gamma)$. By item (6) Lemma E.3, $|\text{CL}^-(\gamma)| = 1$ and therefore $\text{CL}^-(\gamma) = \{\delta'\}$ and $(\delta, \beta] = \text{CL}^-(\gamma) \sqcup [\gamma, \beta] = \{\delta'\} \sqcup [\gamma, \beta]$.

■

The list of intervals $[\delta, \delta']$ with $\text{Cap}[\delta, \delta'] = 2$

$$\begin{aligned}
[\delta, (12)^r], \delta &\in \{(35)^{r-1}, (25)^{r-1}, (15)^{r-1}\}, \\
[\delta, (4)^r], \delta &\in \{(24)^r, (23)^r, (1)^{r-1}\}, \\
[\delta, (25)^r], \delta &\in \{(13)^r, (12)^r, (0)^r\}, \\
[\delta, (23)^r], \delta &\in \{(3)^{r-1}, (4)^{r-1}, (5)^{r-1}\},
\end{aligned} \tag{E.6}$$

In this group of intervals $\{[\delta, \delta']\} \quad \delta' \in \mathbf{M}_2^-$.

$$\begin{aligned}
[(2)^r, \delta'], \delta' &\in \{(24)^{r+1}, (34)^{r+1}, (5)^{r+1}\}, \\
[(45)^r, \delta'], \delta' &\in \{(13)^{r+1}, (14)^{r+1}, (15)^{r+1}\}, \\
[(34)^r, \delta'], \delta' &\in \{(3)^r, (2)^r, (1)^r\}, \\
[(14)^r, \delta'], \delta' &\in \{(35)^r, (45)^r, (0)^{r+1}\},
\end{aligned} \tag{E.7}$$

In this group of intervals $\{[\delta, \delta']\} \quad \delta \in \mathbf{M}_2^+$.

Dualizing modules for non Gorenstein $A[\delta, \delta']$ In some rare cases I have to deal with Cohen-Macaulay algebras $A[\delta, \beta]$, which are not Gorenstein. In particular I am interested in the cases

$$\delta \in \mathbf{M}_2^+, \beta \in \mathbf{M}_1^- \sqcup \mathbf{M}_3^-, \tag{E.8}$$

$$\delta \in \mathbf{M}_1^+ \sqcup \mathbf{M}_3^+, \beta \in \mathbf{M}_2^-. \tag{E.9}$$

Here is an explicit description of the dualizing modules $\omega[\delta, \beta]$:

Proposition E.10 1. In case (E.8) $\omega[\delta, \beta] \cong (\lambda^\gamma, \lambda^{\gamma'}) \subset A[\delta, \beta]$, where $\gamma = \delta$.

γ' is characterized by the condition $\delta \lessdot \gamma' \in \mathbf{M}_3^+$

2. In case (E.9) $\omega[\delta, \beta] \cong (\lambda^\gamma, \lambda^{\gamma'}) \subset A[\delta, \beta]$, where $\gamma = \beta$. γ' is characterized by the condition $\beta \succ \gamma' \in \mathbf{M}_3^-$

Proof. I will only the first statement. By Lemma E.3 item 6 there is a unique $\delta' \in M_1^-, \rho(\delta') = \rho(\delta)$. I define $\delta'' := \delta \wedge \delta' \in M_1^+$. By Proposition 2.47, and (4.28) $A[\delta, \beta]$ and $A(\delta'', \beta)$ are Gorenstein. Straightening relations in $A(\delta'', \beta)$ imply that there is a short exact sequence

$$\{0\} \rightarrow A[\delta', \beta] \xrightarrow{\lambda^\delta \times ?} A(\delta'', \beta) \rightarrow A[\delta, \beta] \rightarrow \{0\} \quad (\text{E.11})$$

$\mathbb{C}(\delta'', \beta]$ -modules. it is explained in the proof of the second part of Lemma 4.24 that $\lambda^\gamma, \lambda^{\gamma'}$ generate $\text{Ann}(\delta')$. The assertion follows from Proposition 3.10. ■

Proposition E.12 *Suppose that $[\delta, \beta]$ satisfies conditions (E.8) or (E.9). Then*

1. $H_{\mathfrak{m}}^i[\delta, \beta] = \{0\}, i \neq \dim = \rho(\beta) - \rho(\delta) + 1$. If $i = \dim$, then there is a short exact sequence

$$\{0\} \rightarrow H_{\mathfrak{m}}^{\dim} A[\delta', \beta] \xrightarrow{\lambda^\delta \times ?} H_{\mathfrak{m}}^{\dim} A(\delta'', \beta) \rightarrow H_{\mathfrak{m}}^{\dim} A[\delta, \beta] \rightarrow \{0\} \quad (\text{E.13})$$

2. $H_{\mathfrak{m}}^i \omega[\delta, \beta] = \{0\}, i \neq 3, \dim$. If $i = \dim$ then $H_{\mathfrak{m}}^{\dim} \omega[\delta, \beta] = H_{\mathfrak{m}}^{\dim}[\delta, \beta]$. If $i = 3$, then $H_{\mathfrak{m}}^3 \omega[\delta, \beta] = H_{\mathfrak{m}}^2 \mathbb{C}[\gamma, \gamma']$.
3. If (E.9) is satisfied, then $H_{\mathfrak{a}}^i((\gamma, \gamma')) \cong H_{\mathfrak{a}}^i \omega[\delta, \beta] \cong H_{\mathfrak{a}}^i[\delta, \beta], i \neq 0. H_{\mathfrak{a}}^3 \omega[\delta, \beta] \cong H_{\mathfrak{m}}^2 \mathbb{C}[\gamma, \gamma']$
4. If (E.8) is satisfied, then $H_{\mathfrak{b}}^i((\gamma, \gamma')) \cong H_{\mathfrak{b}}^i \omega[\delta, \beta] \cong H_{\mathfrak{b}}^i[\delta, \beta], i \neq 0. H_{\mathfrak{b}}^3 \omega[\delta, \beta] \cong H_{\mathfrak{m}}^2 \mathbb{C}[\gamma, \gamma']$

Proof.

The vanishing result from item 1 follows from Lemma 4.13 and Th 16.6 [34]. With a help of Theorem 7.11 [29] I identify (E.13) with a segment of the long exact sequence of local cohomology corresponding to (E.11). Its exactness follows from the vanishing result.

Item 2 follows from consideration of the short exact sequence

$$\{0\} \rightarrow (\lambda^\gamma, \lambda^{\gamma'}) \rightarrow A[\delta, \beta] \rightarrow \mathbb{C}[\gamma, \gamma'] \rightarrow \{0\}$$

and associated long exact sequence of local cohomology.

Last two items are proved similarly. ■

Proposition E.14 *In the assumptions of Proposition E.10*

1. $H_{\mathfrak{m}}^{\dim} \omega[\delta, (1)^{-1}] = H_{\mathfrak{m}}^{\dim}[\delta, (1)^{-1}]$ is a positive energy \mathbf{T} -space with weights bounded from below by $u[\delta'', (1)^{-1}]$ (4.45)
2. $H_{\mathfrak{m}}^{\dim} \omega[(0)^0, \beta] = H_{\mathfrak{m}}^{\dim}[(0)^0, \beta]$ is a negative energy \mathbf{T} -space with weights bounded from above by $u[(0)^0, \beta'']$

Proof. The proof follows from Lemma 4.141 item 3 and Proposition E.12. ■

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